PART III ESSAY: $H \to \gamma\gamma$

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Abstract. The calculation of the leading order decay amplitude $H \to \gamma\gamma$ via a W-loop is made in unitary gauge, initially with dimensional regularisation. The result is compared to Gastmans, Wu and Wu's result of last August, and the discrepancy studied using several methods of computing the final integrals in the calculation, in 't Hooft-Feynman gauge, cutoff regularisation, and loop regularisation. An identity satisfied by a general gauge-invariant regularisation is noted, which permits evaluation of the final amplitude without using a particular regularisation. The difference in the results is traced to the lack of gauge-invariance in Gastmans, Wu and Wu's result, combined with their erroneous use of Dyson subtraction.

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1. Introduction

The Higgs boson is currently a paramount topic in theoretical physics, being subject to the scrutiny of both CERN and the general public. In August of last year, Gastmans, Wu and Wu (henceforth referred to as GWW) calculated the $H \to \gamma\gamma$ decay amplitude using a radically different technique from the original calculations, [4]. Their papers [5, 6] discuss the radically different result they obtained, and attempt to justify its validity using physical arguments.

If this new calculation was correct, there would be far-reaching consequences for both the theoretical and experimental physics of today:

- The $H \to \gamma\gamma$ channel is one of the most important tests of the existence of the Higgs boson. The LHC continues to gather data to improve on the results of last December, [2, 3], and the difference in the GWW amplitude and the standard one causes a significant difference in the frequency of events, which is beyond the scope of this essay to investigate.
- If it were shown that there exist examples where dimensional regularisation does not yield the correct amplitudes because of discontinuity in dimension of the integrals involved, many of the thousands of previous calculations done in the last 50 years using this method would have to be re-scrutinised using more complex regularisation schemata. This would obviously be a tremendous amount of work computationally, as well as requiring a thorough review of the theory involved.

A number of groups subsequently investigated the exact causes of the discrepancy, and a summary is found in [9]. Here we will redo the calculation using a combination of the techniques presented so far, and explain why the GWW calculation is incorrect.

2. Summary

For a number of years there has been discussion of the validity of the technique of dimensional regularisation in QFT calculations: may we assume that all the quantities (chiefly integrals) that we are concerned with are continuous in the dimension?[1]

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[1]There is a reasonable argument that the dimension is a discreet quantity, for one thing. The philosophy of "continuity in dimension" is only really applicable when we replace the integral by a combination of Gaussian integral, and then assume that the generalisation of those integration result to $n = \alpha \notin \mathbb{N}$ dimensions.

GWW argue that their calculation is a demonstration of a result that is not continuous in dimension. A number of papers have shown that in fact their calculation is in error, essentially because their regularisation is not consistent with the gauge, making the result non-gauge invariant.

3. Diagrams and the full amplitude

We will repeat the calculation in unitary gauge, in order to find the controversial steps in GWW. There are only three diagrams in this gauge (shown in Figure 1, reprinted from [6]), because the unphysical field degrees of freedom disappear from the Lagrangian in the limit that produces unitary gauge (as the renormalisation parameter $\xi \to \infty$), and ghost factors can be integrated out directly prior to deriving the Feynman rules. This is the main advantage of unitary gauge, and is one of GWW's main reasons for considering their calculation "simple".

The pertinent Feynman rules are shown in Figure 2, also reprinted from [6]. Here some of the problems of unitary gauge become apparent:

(1) The W propagator does not decay for large values of the momentum p^{α}, so the calculations involve UV divergences in the integrals.
(2) The W propagator has 2 terms. This is in itself not bad, but since there are three of them in two of the diagrams, the integrals we have to evaluate for the decay amplitude contain many terms (72 in the case of $\mathcal{M}_1$!). Obviously this complication increases the chances of algebraic error, but since there are 11 diagrams in the standard computation [4], the author feels that the relative simplicity is a matter of debate.[2]
(3) Because the integrals we will evaluate are nominally divergent, the diagrams have to be summed before manipulation is carried out. The characteristics of the divergent integrals prohibit shifting the integration variables relative to each other for different diagrams, because this introduces spurious terms from non-cancellation.

Therefore we must choose the momentum integration variables consistently. Since $\mathcal{M}_1$ and $\mathcal{M}_3$ are identical apart from swapping the photons, we would like to use this near-symmetry to avoid (even) more calculation.

Since $\mathcal{M}_2$ is totally symmetric in the photons, we can take advantage of this and route the external momenta symmetrically. The sign of the integration variable is irrelevant since only even terms will contribute to the infinite integrals.

We can obtain $\mathcal{M}_2$ from $\mathcal{M}_1$ by shrinking the W propagator between the two photon vertices. Therefore the external momentum routed through the top W in both should be the same.

We have arrived at the choices of GWW. A more detailed explanation is found in [5]. We must add all three diagrams together before we attempt any integration due to the divergence issues discussed earlier.

3.1. Useful bookkeeping for later.

We will make note here of some of the properties of the system that are vital for reducing the size of the calculation later. First, the obvious facts: for the two photons, we have

$$k_i^2 = 0, \qquad k_i \cdot \epsilon(k_i) = 0, \tag{3.1}$$

where $\epsilon(k_i)$ is the polarisation of the ith photon, which we will omit throughout. Because we are omitting this, we can simply declare that $k_{1\mu} = k_{2\nu} = 0$, *for those specific indices.* Through conservation of momentum, the Higgs particle has momentum $k_1 + k_2$. Therefore we have the additional useful appearance of the Higgs mass,

$$M_H^2 = (k_1 + k_2)^2 = 2k_1 \cdot k_2. \tag{3.2}$$

The greater part of [6] is based on eliminating most of the terms containing negative powers of the W mass, which for brevity we will denote by M. This is done by very heavy use of certain

[2]It will be appreciated that both calculations are certainly simpler than the general R_ξ-gauge calculation investigated in [8], however.

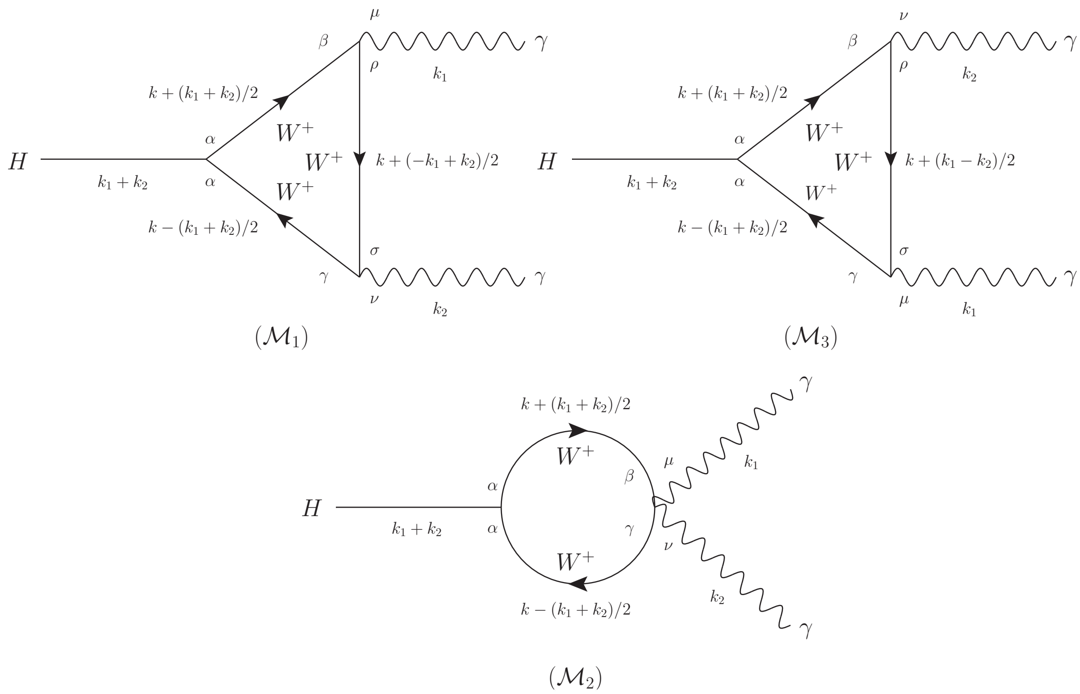


FIGURE 1. The three diagrams of unitary gauge

properties of the $WW\gamma$ vertex, which enables an impressive array of cancellations to take place. We start with a definition of the vertex function:

$$V_{\alpha\beta\gamma}(p_1, p_2, p_3) = (p_2 - p_3)_\alpha g_{\beta\gamma} + (p_3 - p_1)_\beta g_{\gamma\alpha} + (p_1 - p_2)_\gamma g_{\alpha\beta}, \tag{3.3}$$

where the momenta are as shown in the Feynman rules and there is also the conservation of momentum $p_1 + p_2 + p_3 = 0$. We will now contract with the incoming momenta to obtain the following useful identities:

$$\begin{aligned} p_1^\alpha V_{\alpha\beta\gamma}(p_1, p_2, p_3) &= -(p_2 + p_3)^\alpha (p_2 + p_3)_\alpha + (p_3 - p_1)_\beta p_{1\gamma} + (p_1 - p_2)_\gamma p_{1\beta} \\ &= (p_3^2 - p_2^2) g_{\beta\gamma} + p_{3\beta} p_{1\gamma} - p_{2\gamma} p_{1\beta} \\ &= [p_3^2 g_{\beta\gamma} - p_{3\beta} p_{3\gamma}] - [p_2^2 g_{\beta\gamma} - p_{2\beta} p_{2\gamma}]. \end{aligned}$$

When we also include that p_2 is $-k_1$, an outgoing photon momentum, and that we will be contracting with the associated polarisation vector, we lose all the p_2 terms:

$$p_1^\alpha V_{\alpha\mu\gamma}(p_1, -k_1, p_3) = p_3^2 g_{\mu\gamma} - p_{3\mu} p_{3\gamma}. \tag{3.4}$$

This looks more interesting when we rewrite it in the form

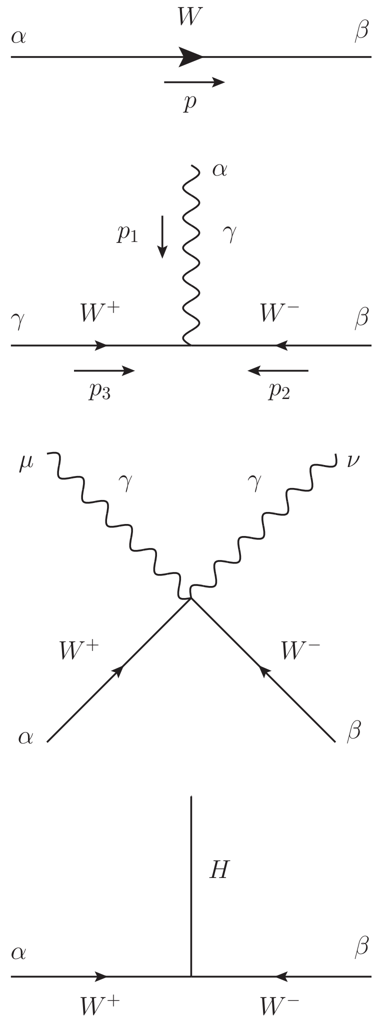
$$p_1^\alpha V_{\alpha\mu\gamma}(p_1, -k_1, p_3) = (p_3^2 - M^2) g_{\mu\gamma} - p_{3\mu} p_{3\gamma} + M^2 g_{\mu\gamma}, \tag{3.5}$$

because the second term is very similar to the W propagator's numerator, and the first is much like the denominator. With this, terms will be moved between orders in M.

It is easy to see from the last equation that contracting with p_3 gives us the remaining identities,

$$p_1^\alpha p_3^\gamma V_{\alpha\mu\gamma}(p_1, -k_1, p_3) = 0, \quad p_1^\alpha p_3^\gamma V_{\alpha\nu\gamma}(p_1, -k_2, p_3) = 0. \tag{3.6}$$

With all the preliminaries out of the way, we can now write down the amplitude, and evaluate it.



$$\frac{i}{p^2 - M^2 + i\epsilon}\left[-g^{\alpha\beta} + \frac{p^\alpha p^\beta}{M^2}\right]$$

$$ie\left[\,(p_2 - p_3)_\alpha\, g_{\beta\gamma} + (p_3 - p_1)_\beta\, g_{\gamma\alpha} + (p_1 - p_2)_\gamma\, g_{\alpha\beta}\,\right]$$

$$-ie^2\left[\,2\, g_{\alpha\beta}\, g_{\mu\nu} - g_{\alpha\mu}\, g_{\beta\nu} - g_{\alpha\nu}\, g_{\beta\mu}\,\right]$$

$$ig\, M\, g_{\alpha\beta}$$

FIGURE 2. The relevant Feynman rules

3.2. **The initial expression.** We write

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3, \tag{3.7}$$

where the right-hand side $\mathcal{M}_i$ are defined in Figure 1.

We reprint here the initial expressions from GWW, which are obtained from the 3 diagrams in the usual way:

$$\mathcal{M}_1 = \frac{-ie^2 gM}{(2\pi)^4} \int d^4k \left[g^{\beta}_{\alpha} - (k + \frac{k_1 + k_2}{2})_\alpha (k + \frac{k_1 + k_2}{2})^\beta / M^2 \right]$$
$$\times \left[g^{\rho\sigma} - (k + \frac{-k_1 + k_2}{2})^\rho (k + \frac{-k_1 + k_2}{2})^\sigma / M^2 \right]$$
$$\times \left[g^{\alpha\gamma} - (k - \frac{k_1 + k_2}{2})^\alpha (k - \frac{k_1 + k_2}{2})^\gamma / M^2 \right]$$
$$\times \left[(k + \frac{3k_1 + k_2}{2})_\rho g_{\beta\mu} + (k + \frac{-3k_1 + k_2}{2})_\beta g_{\mu\rho} + (-2k - k_2)_\mu g_{\rho\beta} \right]$$
$$\times \frac{(k - \frac{k_1+3k_2}{2})_\sigma g_{\gamma\nu} + (k + \frac{-k_1+3k_2}{2})_\gamma g_{\nu\sigma} + (-2k + k_1)_\nu g_{\sigma\gamma}}{\left[\left(k + \frac{k_1+k_2}{2}\right)^2 - M^2 + i\epsilon \right] \left[\left(k + \frac{-k_1+k_2}{2}\right)^2 - M^2 + i\epsilon \right] \left[\left(k - \frac{k_1+k_2}{2}\right)^2 - M^2 + i\epsilon \right]},$$

$$\mathcal{M}_2 = \frac{ie^2 gM}{(2\pi)^4} \int d^4k \left[g^{\beta}_{\alpha} - (k + \frac{k_1 + k_2}{2})_\alpha (k + \frac{k_1 + k_2}{2})^\beta / M^2 \right]$$
$$\times \left[g^{\alpha\gamma} - (k - \frac{k_1 + k_2}{2})^\alpha (k - \frac{k_1 + k_2}{2})^\gamma / M^2 \right]$$
$$\times \frac{2g_{\mu\nu}\, g_{\beta\gamma} - g_{\mu\beta}\, g_{\nu\gamma} - g_{\mu\gamma}\, g_{\nu\beta}}{\left[\left(k + \frac{k_1+k_2}{2}\right)^2 - M^2 + i\epsilon \right] \left[\left(k - \frac{k_1+k_2}{2}\right)^2 - M^2 + i\epsilon \right]},$$

$$\mathcal{M}_3 = \frac{-ie^2 gM}{(2\pi)^4} \int d^4k \left[g^{\beta}_{\alpha} - (k + \frac{k_1 + k_2}{2})_\alpha (k + \frac{k_1 + k_2}{2})^\beta / M^2 \right]$$
$$\times \left[g^{\rho\sigma} - (k + \frac{k_1 - k_2}{2})^\rho (k + \frac{k_1 - k_2}{2})^\sigma / M^2 \right]$$
$$\times \left[g^{\alpha\gamma} - (k - \frac{k_1 + k_2}{2})^\alpha (k - \frac{k_1 + k_2}{2})^\gamma / M^2 \right]$$
$$\times \left[(k + \frac{k_1 + 3k_2}{2})_\rho g_{\beta\nu} + (k + \frac{k_1 - 3k_2}{2})_\beta g_{\nu\rho} + (-2k - k_1)_\nu g_{\rho\beta} \right]$$
$$\times \frac{(k - \frac{3k_1+k_2}{2})_\sigma g_{\gamma\mu} + (k + \frac{3k_1-k_2}{2})_\gamma g_{\mu\sigma} + (-2k + k_2)_\mu g_{\sigma\gamma}}{\left[\left(k + \frac{k_1+k_2}{2}\right)^2 - M^2 + i\epsilon \right] \left[\left(k + \frac{k_1-k_2}{2}\right)^2 - M^2 + i\epsilon \right] \left[\left(k - \frac{k_1+k_2}{2}\right)^2 - M^2 + i\epsilon \right]}.$$

We have the usual notation, e as the unit charge, g the electroweak coupling, M the W mass. We have omitted the polarisations, as previously mentioned. The tensor indices on the $\mathcal{M}_i$ will be omitted unless we specifically wish to draw attention to them.

At the moment the algebra looks horrible, but we shall now remove most of the terms. For the time being, since we will not do any integration for a while, we shall set

$$p = k + \frac{k_1 + k_2}{2} \quad q = k + \frac{-k_1 + k_2}{2} \quad r = k - \frac{k_1 + k_2}{2},$$

which makes the amplitudes shorter at least: for example, $\mathcal{M}_1$ becomes

$$\mathcal{M}_1 = \frac{-ie^2 gM}{(2\pi)^4} \int d^4k \left[g^{\beta}_{\alpha} - p_\alpha p^\beta / M^2 \right] \left[g^{\rho\sigma} - q^\rho q^\sigma / M^2 \right] \left[g^{\alpha\gamma} - r^\alpha r^\gamma / M^2 \right]$$
$$\times \frac{V_{\mu\rho\beta}(-k_1, -q, p) V_{\nu\gamma\sigma}(-k_2, -r, q)}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]}$$

and $\mathcal{M}_2$ is

$$\mathcal{M}_2 = \frac{ie^2 gM}{(2\pi)^4} \int d^4k \left[g^{\beta}_{\alpha} - p_\alpha p^\beta / M^2 \right] \left[g^{\alpha\gamma} - r^\alpha r^\gamma / M^2 \right] \frac{2g_{\mu\nu}\, g_{\beta\gamma} - g_{\mu\beta}\, g_{\nu\gamma} - g_{\mu\gamma}\, g_{\nu\beta}}{[p^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]}$$

It will be seen that this enables more economy of expression than [6]'s notation, although it is not useful for the entire calculation.

4. Simplifying the amplitude expression

4.1. Negative powers of M.

4.1.1. M^{-6}. This term is particularly simple. We can see that the only contribution to this will come from the terms in $\mathcal{M}_1$; and if we can remove this, then $\mathcal{M}_3$ will work in exactly the same way. The term in question contains

$$\begin{aligned} &-p_\alpha p^\beta q^\rho q^\sigma r^\alpha r^\gamma V_{\mu\rho\beta}(-k_1,-q,p) V_{\nu\gamma\sigma}(-k_2,-r,q) \\ &= -(p\cdot r)\left[p^\beta q^\rho V_{\mu\rho\beta}(-k_1,-q,p)\right]\left[q^\sigma r^\gamma V_{\nu\gamma\sigma}(-k_2,-r,q)\right]. \end{aligned}$$

It is immediately obvious that this is in the form of the identities (3.6), so there are no M^{-6} terms at all.

4.1.2. M^{-4}. This is where things become somewhat trickier. First look at the 3 terms in $\mathcal{M}_1$:

$$-\left(q^\rho q^\sigma r^\beta r^\gamma + g^{\rho\sigma}(p\cdot r)p^\beta r^\gamma + p^\gamma p^\beta q^\rho q^\sigma\right) V_{\mu\rho\beta}(-k_1,-q,p) V_{\nu\gamma\sigma}(-k_2,-r,q)$$

Clearly the identities (3.6) once again remove a couple of terms, so we are only dealing with the middle term.

The term from $\mathcal{M}_2$ is proportional to

$$p_\alpha p^\beta r^\alpha r^\gamma \left(2g_{\mu\nu}\, g_{\beta\gamma} - g_{\mu\beta}\, g_{\nu\gamma} - g_{\mu\gamma}\, g_{\nu\beta}\right) = (p\cdot r)\left(g_{\mu\nu}(p\cdot r) - p_\mu r_\nu - r_\mu p_\nu\right) \tag{4.1}$$

We will now work only with the integrands to save space, and we can drop the factor of $(p\cdot r)$ temporarily since it appears everywhere. The reduced integrand of $\mathcal{M}_1$ is therefore

$$A_{11} = -\frac{g^{\rho\sigma} p^\beta r^\gamma V_{\mu\rho\beta}(-k_1,-q,p) V_{\nu\gamma\sigma}(-k_2,-r,q)}{[p^2-M^2+i\epsilon]\,[q^2-M^2+i\epsilon]\,[r^2-M^2+i\epsilon]} \tag{4.2}$$

We now apply (3.5), repeatedly, to split the integral until we can deal with the terms. A_{11} contains

$$p^\beta V_{\mu\rho\beta}(-k_1,-q,p) = (q^2-M^2) g_{\mu\rho} - q_\mu q_\rho + M^2 g_{\mu\rho},$$

so it splits as

$$A_{11} = A_{111} + A_{112} + A_{113},$$

where (following the subscripting of [6])

$$\begin{aligned} A_{111} &= -\frac{g^{\rho\sigma} r^\gamma g_{\mu\rho} V_{\nu\gamma\sigma}(-k_2,-r,q)}{[p^2-M^2+i\epsilon]\,[r^2-M^2+i\epsilon]} \\ &= -\frac{r^\gamma V_{\nu\gamma\mu}(-k_2,-r,q)}{[p^2-M^2+i\epsilon]\,[r^2-M^2+i\epsilon]} \end{aligned} \tag{4.3}$$

$$\begin{aligned} A_{112} &= +\frac{g^{\rho\sigma} r^\gamma q_\mu q_\rho V_{\nu\gamma\sigma}(-k_2,-r,q)}{[p^2-M^2+i\epsilon]\,[q^2-M^2+i\epsilon]\,[r^2-M^2+i\epsilon]} \\ &= \frac{r^\gamma q_\mu q^\sigma V_{\nu\gamma\sigma}(-k_2,-r,q)}{[p^2-M^2+i\epsilon]\,[q^2-M^2+i\epsilon]\,[r^2-M^2+i\epsilon]} \\ &= 0 \end{aligned}$$

by using (3.6), this term is immediately 0. We are not so lucky with A_{113}:

$$\begin{aligned} A_{113} &= M^2 \frac{g^{\rho\sigma} r^\gamma g_{\mu\rho} V_{\nu\gamma\sigma}(-k_2,-r,q)}{[p^2-M^2+i\epsilon]\,[q^2-M^2+i\epsilon]\,[r^2-M^2+i\epsilon]} \\ &= M^2 \frac{r^\gamma V_{\nu\gamma\mu}(-k_2,-r,q)}{[p^2-M^2+i\epsilon]\,[q^2-M^2+i\epsilon]\,[r^2-M^2+i\epsilon]} \end{aligned} \tag{4.4}$$

so we will have to look at this term at the same time as the M^{-2} terms in the next section.

To finish off the terms at this order, use (3.4) on A_{111}. We find that the numerator is

$$-q^2 g_{\mu\nu} + q_\mu q_\nu,$$

where there is a difference in sign from the order of the arguments in V. We now go back to the original variables because it is easier to switch between $\mathcal{M}_1$ and $\mathcal{M}_3$. Clearly there are 2 separate terms to look at: one proportional to $g_{\mu\nu}$ and the other a collection of k_i-built tensors.

For the $g_{\mu\nu}$ part, we expand

$$q^2 = k^2 + (-k_1 + k_2) \cdot k + (-k_1 + k_2)^2/4 = k^2 + (-k_1 + k_2) \cdot k - k_1 \cdot k_2/2.$$

The transformation taking $\mathcal{M}_1 \leftrightarrow \mathcal{M}_3$ is swapping the photons, i.e. $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$. Since the linear term is antisymmetric under the transformation, it cancels in the total $\mathcal{M}$, and the contribution we are left with is

$$-2k^2 + k_1 \cdot k_2.$$

From equation (4.1), the term at this order from $\mathcal{M}_{21}$ contributes

$$2(p \cdot r) = 2k^2 - (k_1 + k_2)^2/2 = 2k^2 - k_1 \cdot k_2,$$

and we see the terms sum to 0.

Similarly,

$$q_\mu q_\nu = (k + k_2/2)_\mu (k - k_1/2)_\nu,$$

which is equal to $p_\mu r_\nu$, and swapping the k_i and indices, we find a term equal to $p_\nu r_\mu$, which we see cancels the other terms in 4.1.

Hence there are no terms at order M^{-4}.

4.1.3. M^{-2}. This part is considerably worse than the last two orders. There are 3 terms that come straight from $\mathcal{M}_1$, 2 more from $\mathcal{M}_2$, and we still have A_{113} floating around at this order.

The three terms from $\mathcal{M}_1$:

$$\mathcal{M}_{12} = \frac{ie^2 gM}{(2\pi)^4 M^2} \int d^4k \frac{g^{\rho\sigma} r^\beta r^\gamma V_{\mu\rho\beta}(-k_1, -q, p) V_{\nu\gamma\sigma}(-k_2, -r, q)}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.5}$$

$$\mathcal{M}_{13} = \frac{ie^2 gM}{(2\pi)^4 M^2} \int d^4k \frac{q^\rho q^\sigma g^{\beta\gamma} V_{\mu\rho\beta}(-k_1, -q, p) V_{\nu\gamma\sigma}(-k_2, -r, q)}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.6}$$

$$\mathcal{M}_{14} = \frac{ie^2 gM}{(2\pi)^4 M^2} \int d^4k \frac{p^\gamma p^\beta g^{\rho\sigma} V_{\mu\rho\beta}(-k_1, -q, p) V_{\nu\gamma\sigma}(-k_2, -r, q)}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.7}$$

The two terms from $\mathcal{M}_2$:

$$\mathcal{M}_{22} = \frac{-ie^2 gM}{(2\pi)^4 M^2} \int d^4k \frac{p^\gamma p^\beta \left(2g_{\mu\nu}\, g_{\beta\gamma} - g_{\mu\beta}\, g_{\nu\gamma} - g_{\mu\gamma}\, g_{\nu\beta}\right)}{[p^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.8}$$

$$= \frac{-ie^2 gM}{(2\pi)^4 M^2} \int d^4k \frac{2p^2 g_{\mu\nu} - 2p_\mu p_\nu}{[p^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.9}$$

$$\mathcal{M}_{23} = \frac{-ie^2 gM}{(2\pi)^4 M^2} \int d^4k \frac{r^\beta r^\gamma \left(2g_{\mu\nu}\, g_{\beta\gamma} - g_{\mu\beta}\, g_{\nu\gamma} - g_{\mu\gamma}\, g_{\nu\beta}\right)}{[p^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.10}$$

$$= \frac{-ie^2 gM}{(2\pi)^4 M^2} \int d^4k \frac{2r^2 g_{\mu\nu} - 2r_\mu r_\nu}{[p^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.11}$$

Now, we repeat the application of (3.5) from before. We can use it on $\mathcal{M}_{12}$ to cancel parts with $\mathcal{M}_{23}$ in the same way as before; we are left with the terms

$$\mathcal{M}_{122} = \frac{-ie^2 gM}{(2\pi)^4 M^2} \int d^4k \frac{r^\beta q^\rho q_\nu V_{\mu\rho\beta}(-k_1, -q, p)}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.12}$$

$$\mathcal{M}_{123} = \frac{ie^2 gM}{(2\pi)^4} \int d^4k \frac{r^\beta V_{\mu\nu\beta}(-k_1, -q, p)}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]}, \tag{4.13}$$

and the latter moves up to contribute to the answer. Similarly, the $\mathcal{M}_{14}$ terms left are

$$\mathcal{M}_{142} = \frac{-ie^2 gM}{(2\pi)^4 M^2} \int d^4k \frac{q_\mu q^\sigma p^\gamma V_{\nu\gamma\sigma}(-k_2, -r, q)}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.14}$$

$$\mathcal{M}_{143} = \frac{ie^2 gM}{(2\pi)^4} \int d^4k \frac{p^\gamma V_{\nu\gamma\mu}(-k_2, -r, q)}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]}, \tag{4.15}$$

The remaining terms at order M^{-2} are therefore $\mathcal{M}_{113}$, $\mathcal{M}_{13}$, $\mathcal{M}_{122}$ and $\mathcal{M}_{142}$. Collecting their numerators, the integrand is proportional to

$$\begin{aligned} A = {} & (p\cdot r) r^\gamma V_{\nu\gamma\mu}(-k_2, -r, q) - q^\rho q^\sigma g^{\beta\gamma} V_{\mu\rho\beta}(-k_1, -q, p) V_{\nu\gamma\sigma}(-k_2, -r, q) \\ & + r^\beta q^\rho q_\nu V_{\mu\rho\beta}(-k_1, -q, p) + q_\mu q^\sigma p^\gamma V_{\nu\gamma\sigma}(-k_2, -r, q) \end{aligned}$$

Every one of these is amenable to (3.4) at least once. The expression reduces to

$$\begin{aligned} A = {} & \left[(p\cdot r)\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\right] + \left[-p^2 r^2 g_{\mu\nu} + r^2 p_\mu p_\nu - (p\cdot r) p_\mu r_\nu\right] \\ & + \left[p^2 r_\mu q_\nu - (p\cdot r) p_\mu q_\nu\right] + \left[r^2 q_\mu p_\nu - (p\cdot r) q_\mu r_\nu\right], \end{aligned} \tag{4.16}$$

which corresponds to GWW's (3.36).

Several pages of algebra enable us to obtain their (3.37) by expanding brackets and cancelling many of the terms,[3]

$$\begin{aligned} A = {} & 4(k_1\cdot k_2) k_\mu k_\nu + 2k^2 k_{2\mu} k_{1\nu} - 2(k_\mu k_{1\nu} + k_{2\mu} k_\nu (k_1 + k_2)\cdot k \\ & + g_{\mu\nu}\left[-2k^2 (k_1\cdot k_2) + ((k_1 + k_2)\cdot k)\right] \\ & + \left[k^2 - \frac{k_1\cdot k_2}{2}\right]\left[-g_{\mu\nu}(k_1 + k_2)\cdot k + 2(k_\mu k_{1\nu} - k_{2\mu} k_\nu)\right] \end{aligned} \tag{4.17}$$

Finally, we note that since

$$q^2 = k^2 - (k_1 - k_2)\cdot k - (k_1\cdot k_2)/2,$$

we can manipulate the last term to obtain two pieces:

$$B_1 + B_2 = \left[k^2 - (k_1 - k_2)\cdot k - (k_1\cdot k_2)/2 - M^2\right]\left[-g_{\mu\nu}(k_1 + k_2)\cdot k + 2(k_\mu k_{1\nu} - k_{2\mu} k_\nu)\right] \tag{4.18}$$

$$+ \left[(k_1 - k_2) + M^2\right]\left[-g_{\mu\nu}(k_1 + k_2)\cdot k + 2(k_\mu k_{1\nu} - k_{2\mu} k_\nu)\right] \tag{4.19}$$

B_1 contains the factor which cancels the q bracket in the denominator of the integral. We note that the denominator will now contain only even powers of k, by a simple expansion. But $\left[-g_{\mu\nu}(k_1 + k_2)\cdot k + 2(k_\mu k_{1\nu} - k_{2\mu} k_\nu)\right]$ contains only odd powers of k, so we conclude that the integral involving B_1 is 0.[4] B_2 leaves us 2 more terms: one that joins the others at this order, and one which moves up to the final order, M^0. They are:

$$\mathcal{M}_{1131} = \frac{-ie^2 gM}{(2\pi)^4} \frac{1}{M^2} \int d^4k \times \frac{4(k_1\cdot k_2) k_\mu k_\nu + 2k^2 k_{2\mu} k_{1\nu} - 4k_\mu k_{1\nu}(k\cdot k_2) - 4k_{2\mu} k_\nu (k\cdot k_1) + g_{\mu\nu}(-2k^2(k_1\cdot k_2) + 4(k\cdot k_1)(k\cdot k_2))}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.20}$$

and

$$\mathcal{M}_{1132} = \frac{-ie^2 gM}{(2\pi)^4} \int d^4k \frac{-g_{\mu\nu}(k_1 + k_2)\cdot k + 2(k_\mu k_{1\nu} - k_{2\mu} k_\nu)}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.21}$$

We shall actually consider $\mathcal{M}_{1131}$ later, as it is one of the main points of contention in the GWW calculation. We move up to the final level.

---

[3]Doubtless there is a way to reach an equivalent result using our variables, but the algebra is particularly unhelpful at this point.

[4]As pointed out in [9], an integral of the form $\int dl\, l^\mu/$(even function) must vanish because there is no invariant rank-1 tensor it can be proportional to.

4.2. **Everything else.** We now have to collect up everything left from the previous section, and combine it with the final two terms from the two types of diagram. The last contribution from $\mathcal{M}_1$ is

$$\mathcal{M}_{15} = \frac{-ie^2 gM}{(2\pi)^4} \int d^4k \frac{g^{\rho\sigma} g^{\beta\gamma} V_{\mu\rho\beta}(-k_1, -q, p) V_{\nu\gamma\sigma}(-k_2, -r, q)}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]} \tag{4.22}$$

and that from $\mathcal{M}_2$ is

$$\mathcal{M}_{24} = \frac{ie^2 gM}{(2\pi)^4} \int d^4k \frac{g^{\beta\gamma}\left(2g_{\mu\nu}\, g_{\beta\gamma} - g_{\mu\beta}\, g_{\nu\gamma} - g_{\mu\gamma}\, g_{\nu\beta}\right)}{[p^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]}.$$

Since $\mathcal{M}_2$ is symmetric when we interchange the photons, we can write the terms with the overall mass factor M as

$$\mathcal{M}_T = \mathcal{M}_T^{(1)} + \mathcal{M}_T^{(3)},$$

where now we are using the notation of [7]; these two terms are defined by

$$\mathcal{M}_T^{(1)} = \mathcal{M}_{15} + \frac{1}{2}\mathcal{M}_{24} + \mathcal{M}_{123} + \mathcal{M}_{143} + \mathcal{M}_{1132}, \tag{4.23}$$

with $\mathcal{M}_T^{(3)}$ defined similarly in the obvious way. We will now switch to n dimensions in anticipation of using dimensional regularisation at some point; we can set n back to 4 later for further discussion. Similarly, set the other integral we have left to be

$$\mathcal{M}_L^{(1)} = \mathcal{M}_{1131}. \tag{4.24}$$

We then find that the numerator of $\mathcal{M}_{15}$ becomes

$$g_{\mu\nu}\left[2k^2 - (k \cdot k_1) + (k \cdot k_2) - 5(k_1 \cdot k_2)\right] + 2k_\mu k_\nu + \frac{9}{2}k_{2\mu}k_{1\nu} + 4(n-2)(k + k_2/2)_\mu (k - k_1/2)_\nu; \tag{4.25}$$

that of $\mathcal{M}_{24}/2$ is easily seen to be

$$-(n-1)g_{\mu\nu}(k^2 - (k_1 \cdot k) + (k_2 \cdot k) - k_1 \cdot k_2/2 - M^2). \tag{4.26}$$

We now look at $\mathcal{M}_{123}$ and $\mathcal{M}_{143}$. Both have an overall minus sign. The numerators are:

$$\begin{aligned} &g_{\mu\nu}\left[\left(k - \frac{k_1 + k_2}{2}\right) \cdot \left(k + \frac{-3k_1 + k_2}{2}\right)\right] - 2\left(k + \frac{k_2}{2}\right)_\mu \left(k - \frac{k_1}{2}\right)_\nu + \left(k - \frac{k_2}{2}\right)_\mu \left(k + \frac{3k_1}{2}\right)_\nu \\ &= g_{\mu\nu}\left[k^2 - 2(k \cdot k_1) + \frac{k_1 \cdot k_2}{2}\right] - 2\left(k + \frac{k_2}{2}\right)_\mu \left(k - \frac{k_1}{2}\right)_\nu + k_\mu k_\nu - \frac{1}{2}k_{2\mu}k_\nu + \frac{3}{2}k_\mu k_{1\nu} - \frac{3}{4}k_{2\mu}k_{1\nu} \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} &g_{\mu\nu}\left[\left(k + \frac{k_1 + k_2}{2}\right) \cdot \left(k + \frac{-k_1 + 3k_2}{2}\right)\right] - 2\left(k + \frac{k_2}{2}\right)_\mu \left(k - \frac{k_1}{2}\right)_\nu + \left(k - \frac{3k_2}{2}\right)_\mu \left(k + \frac{k_1}{2}\right)_\nu \\ &= g_{\mu\nu}\left[k^2 + 2(k \cdot k_2) + \frac{k_1 \cdot k_2}{2}\right] - 2\left(k + \frac{k_2}{2}\right)_\mu \left(k - \frac{k_1}{2}\right)_\nu + k_\mu k_\nu - \frac{3}{2}k_{2\mu}k_\nu + \frac{1}{2}k_\mu k_{1\nu} - \frac{3}{4}k_{2\mu}k_{1\nu}. \end{aligned} \tag{4.28}$$

Summing these, we find

$$g_{\mu\nu}\left[2k^2 - 2(k \cdot k_1) + 2(k \cdot k_2) + k_1 \cdot k_2\right] - 4\left(k + \frac{k_2}{2}\right)_\mu \left(k - \frac{k_1}{2}\right)_\nu + 2k_\mu k_\nu - 2k_{2\mu}k_\nu + 2k_\mu k_{1\nu} - \frac{3}{2}k_{2\mu}k_{1\nu}$$

Finally, the numerator from $\mathcal{M}_{1132}$:

$$-g_{\mu\nu}\left[(k \cdot k_1) - (k \cdot k_2)\right] + 2(k_\mu k_{1\nu} - k_{2\mu}k_\nu). \tag{4.29}$$

We add everything together and reach

$$\mathcal{M}_T^{(1)} = \frac{-ie^2 gM}{(2\pi)^4} \int d^n k \frac{C}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]}, \tag{4.30}$$

where

$$
\begin{aligned}
C = g_{\mu\nu} & \left[-(n-1)\left(k^2 - (k\cdot k_1) + (k\cdot k_2)\right) + \frac{n-13}{2} k_1 \cdot k_2 \right] \\
& + (n-1)M^2 g_{\mu\nu} \\
& + 4(n-1)k_\mu k_\nu + (7-n)k_{2\mu}k_{1\nu} + 2(n-1)(-k_\mu k_{1\nu} + k_{2\mu}k_\nu)
\end{aligned} \tag{4.31}
$$

We are now have two integrals we need to study.

5. Calculation of the integrals

Both integrals are of the form

$$\mathcal{M}_i = \frac{-ie^2 gM}{(2\pi)^4\{M^2\}} \int d^n k \frac{M_i}{[p^2 - M^2 + i\epsilon]\,[q^2 - M^2 + i\epsilon]\,[r^2 - M^2 + i\epsilon]}, \tag{5.1}$$

where the M^2 is only present in $\mathcal{M}_L^{(1)}$.

We once again follow [6] and use the Feynman parametrisation, which is this case is of the form

$$\frac{1}{ABC} = 2\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \frac{\delta(1-\alpha_1-\alpha_2-\alpha_3)}{(\alpha_1 A + \alpha_2 B + \alpha_3 C)^3} \tag{5.2}$$

$$= 2\int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \frac{1}{(\alpha_1 A + \alpha_2 B + (1-\alpha_1-\alpha_2)C)^3} \tag{5.3}$$

We can perform a simplification on the new denominator D: each bracket contains a $k^2 - M^2 + i\epsilon$, so we only have to change the other 3 types of term:

$$
\begin{aligned}
D &= \alpha_1 p^2 + \alpha_3 q^2 + \alpha_2 r^2 - M^2 + i\epsilon \\
&= k^2 + \alpha_1 k\cdot(k_1+k_2) + \alpha_3 k\cdot(-k_1+k_2) - \alpha_2 k\cdot(k_1+k_2) + (\alpha_1 - \alpha_3 + \alpha_2)\frac{k_1\cdot k_2}{2} - M^2 + i\epsilon \\
&= k^2 - (1-2\alpha_1)(k\cdot k_1) + (1-2\alpha_2)(k\cdot k_2) + (1-2\alpha_3)\frac{k_1\cdot k_2}{2} - M^2 + i\epsilon,
\end{aligned} \tag{5.4}
$$

where we have used $\sum_i \alpha_i = 1$.

We notice that at this point it is very convenient to complete the square on the first 3 terms,

$$D = \left[k - \frac{1}{2}(1-2\alpha_1)k_1 + \frac{1}{2}(1-2\alpha_2)k_2\right]^2 + 2\alpha_1\alpha_2(k_1\cdot k_2) - M^2 + i\epsilon, \tag{5.5}$$

This suggests we should substitute

$$k = l + \frac{1}{2}(1-2\alpha_1)k_1 - \frac{1}{2}(1-2\alpha_2)k_2, \tag{5.6}$$

but we should first ask if this causes problems with so-called “surface terms”: terms which appear when shifting the integration variable in some divergent integrals. Luckily for $n < 4$, both integrals are finite.[5] Therefore we make the substitution into both $\mathcal{M}_T^{(1)}$ and $\mathcal{M}_L^{(1)}$: their denominators become

$$\left[l^2 - M^2 + i\epsilon + 2\alpha_1\alpha_2(k_1\cdot k_2)\right]^3,$$

and after some lengthy algebra, we can simplify the numerators: firstly,

$$M_L^{(1)} = 4(k_1\cdot k_2)l_\mu l_\nu + 2k_{2\mu}k_{1\nu}l^2 - 4(k_2\cdot l)l_\mu k_{1\nu} - 4(k_1\cdot l)k_{2\mu}l_\nu - 2g_{\mu\nu}(k_1\cdot k_2)l^2 + 4g_{\mu\nu}(l\cdot k_1)(l\cdot k_2). \tag{5.7}$$

[5] we can see this by counting powers: k^2 on the top, $(k^2)^3$ on the bottom, n integrals, suggests the integral is convergent since $2 - 6 + n = n - 4 < 0$.

We have dropped all terms with an odd number of l, because their integrations necessarily vanish. All terms that contain no l have also vanished in M_{1131}. Similarly,

$$M_T^{(1)} = -(n-1)g_{\mu\nu}l^2 + 4(n-2)l_\mu l_\nu + g_{\mu\nu}\left[-(k_1\cdot k_2)(6+2(n-1)\alpha_1\alpha_2) + (n-1)M^2\right] + 2(3-2(n-1)\alpha_1\alpha_2)k_{2\mu}k_{1\nu}. \tag{5.8}$$

There are several points that should be noted about the integral of this quantity. The first is that the first line forms the controversial integral which shall be the subject of much of the remainder of the paper. Also, the constant tensor terms have not disappeared this time, and instead form a simple convergent integral which we shall discuss briefly below.

Furthermore, since both of these integrals are now entirely symmetric in the photon properties, there is nothing else we need to compute: we obtain the answer as $\mathcal{M} = 2(\mathcal{M}_L^{(1)} + \mathcal{M}_T^{(1)})$.

5.1. **Final manipulation of the expressions.** We will now convert the integrals into a somewhat simpler form simply by rewriting the numerators yet again, to separate the constants from the integration momentum: first consider $M_L^{(1)}$. Using an idea in [7], we can write it as

$$\mathcal{M}_L^{(1)} = -\left[(k_1\cdot k_2)g_\mu^\rho g_\nu^\sigma - k_2^\sigma k_{1\nu}g_\mu^\rho - k_1^\rho k_{2\mu}g_\nu^\sigma + k_1^\rho k_2^\sigma g_{\mu\nu}\right]\left(l^2 g_{\rho\sigma} - 4l_\rho l_\sigma\right) \tag{5.9}$$

We can carry out a similar process on $M_T^{(1)}$, although it is a case of separating the Feynman variables here:

$$\begin{aligned} M_T^{(1)} &= -(n-1)\left[g_{\mu\nu}l^2 - 4l_\mu l_\nu\right] \\ &\quad + (n-1)\left(M^2 - 2\alpha_1\alpha_2(k_1\cdot k_2)\right)g_{\mu\nu} \\ &\quad + \left[k_{2\mu}k_{1\nu} - g_{\mu\nu}(k_1\cdot k_2)\right](6-4(n-1)\alpha_1\alpha_2)\,. \end{aligned} \tag{5.10}$$

We must now actually calculate the integrals. We start with the simplest method we are prepared for.

5.2. **The finite integral.** We can take $n = 4$ for this integral, since it converges absolutely. It has the form

$$I_{-2} = \int d^4l \frac{1}{[l^2 - M^2 + 2\alpha_1\alpha_2(k_1\cdot k_2) + i\epsilon]^3}$$

After a Wick rotation $l_0 \to i l_0$, we find

$$-i\int d^4l \frac{1}{[l^2 + M^2 - 2\alpha_1\alpha_2(k_1\cdot k_2) - i\epsilon]^3}.$$

Change variables to $r^2 = l^2$, and $d^4l = r^3 dr d\Omega_3$,

$$= -i\frac{4\pi^2}{\Gamma(3)}\int_0^\infty dr \frac{r^3}{[r^2 + M^2 - 2\alpha_1\alpha_2(k_1\cdot k_2) - i\epsilon]^3},$$

using the area of the n-sphere. This integral is simple, and we cover a more general case in the next section, so we find

$$I_{-2} = -\frac{i\pi^2}{2}\frac{1}{M^2 - 2\alpha_1\alpha_2(k_1\cdot k_2) - i\epsilon} \tag{5.11}$$

We can use this in several places in the calculation, leading to

$$-\frac{e^2 gM}{8\pi^2}\left[k_{2\mu}k_{1\nu} - g_{\mu\nu}(k_1\cdot k_2)\right]6\int_0^1 d\alpha_1\int_0^{1-\alpha_1} d\alpha_2 \frac{1-\frac{2}{3}(n-1)\alpha_1\alpha_2}{M^2 - 2\alpha_1\alpha_2(k_1\cdot k_2) - i\epsilon} \tag{5.12}$$

This is essentially GWW's answer, after we take $n = 4$. The double integral is simple enough to do: recalling that $2k_1\cdot k_2 = 2M_H^2$, and defining

$$z = \frac{M_H^2}{4M^2 - i\epsilon},$$

(a slight variant of the definition of τ in most of the other papers), we can write the integrand as

$$\frac{1}{M^2}\frac{1-\frac{2}{3}(n-1)\alpha_1\alpha_2}{1-4z\alpha_1\alpha_2} = \frac{1}{M^2}\left(\frac{1}{2z} + \frac{n-1}{3}\left(\frac{2}{z} - \frac{1}{z^2}\right)\frac{1}{z^{-1}-4\alpha_1\alpha_2}\right).$$

The first term is obviously easy to integrate and contributes a factor of 1/2. For the second term, define

$$J(z) = \int_0^1 dx \int_0^{1-x} dy \frac{1}{z^{-1}-4xy}. \tag{5.13}$$

It is fairly obvious that $J(0)=0$, because the integrand is nonsingular for small $|z|$, and then the integral itself $\sim z/2$ for small z. We will obtain the complete result by differentiation under the integral sign, then choosing the correct integration path for z. We have:

$$\begin{aligned} J'(z) &= \int_0^1 dx \int_0^{1-x} dy \frac{1}{(1-4xyz)^2} \\ &= \int_0^1 dx \frac{1-x}{1-4x(1-x)z} \\ &= \int_0^1 dx \frac{1-x}{4z(x-1/2)^2+(1-z)}. \end{aligned}$$

We then substitute $x - 1/2 = \tan\theta$, etc., and the integral is found to be

$$J'(z) = \frac{\arctan\left(\sqrt{z}/\sqrt{1-z}\right)}{2\sqrt{z(1-z)}}.$$

It is easy enough to see that an antiderivative of this in the neighbourhood of the origin with the correct behaviour for real $0<z<\delta$, for some small $\delta>0$ is

$$J(z) = \frac{1}{2}\arctan^2\frac{\sqrt{z}}{\sqrt{1-z}} = \frac{1}{2}\arcsin^2\sqrt{z},$$

which agrees with the usual result for $|z| \le 1$, at least. The question of which side of the branch point at 1 to pass is solved by considering the form of z:

$$z \sim \frac{M_H^2}{M^2}\left(1+\frac{i\epsilon}{M^2}\right),$$

so z is displaced slightly upwards from the real line and we need to pass above the branch point, which leads to the

$$-\frac{1}{4}\left(\log\frac{1+\sqrt{1-z^{-1}}}{1-\sqrt{1-z^{-1}}} - i\pi\right)^2$$

result. We have reached GWW's result,

$$\mathcal{M}_{\text{finite}} = -\frac{3e^2g}{8\pi^2M}\left[k_{2\mu}k_{1\nu} - g_{\mu\nu}(k_1\cdot k_2)\right]\left[\tau^{-1} + \left(2\tau^{-1}-\tau^{-2}\right)f(\tau)\right]. \tag{5.14}$$

But there are more integrals to examine. First is another finite one, from the second line of equation (5.10). It is clear that the Feynman variable dependence conveniently disappears, so this integral gives us the rather unnerving term

$$\mathcal{M}_g = -\frac{e^2gM}{8\pi^4}\left[\frac{-i\pi^2}{2}(n-1)g_{\mu\nu}\right]. \tag{5.15}$$

The problems with this term as it stands will be explained below in Section 6.

5.3. **Dimensional regularisation.** We have remaining essentially 2 integrals that have the same form, and one very simple one. We change to a Euclidean variable once again, which essentially flips all signs in the denominators that are not on l^2s, and introduces an overall factor of $-i$. GWW's notion [5, 6] is that both of these integrals vanish in $n = 4$ dimensions, essentially because we may use the identity

$$g^{\mu\nu}(g_{\mu\nu} - 4l_\mu l_\nu) = (n-4)l^2$$

with $n = 4$ to cancel both of these integrals. This is not valid in this case, as demonstrated in Section 6.3. We shall now use dimensional regularisation to obtain the correct result for these integrals.

The integral has been reduced to the form

$$I_{\mu\nu} = \int d^4l \frac{l^2 g_{\mu\nu} - 4l_\mu l_\nu}{(a+l^2)^3}, \tag{5.16}$$

where a is the rest of the denominator, $M^2 - 2\alpha_1\alpha_2(k_1 \cdot k_2 - i\epsilon)$.

Since $I_{\mu\nu}$ contains no variables, for it to be an invariant tensor, it must be a sum of invariant tensors of rank 2. Only one of these exists that is correctly invariant, the metric $g_{\mu\nu}$. Therefore if the integral is to be a proper tensor, it is of the form $I_{\mu\nu} = Ig_{\mu\nu}$. Saturating (5.16) with $g^{\mu\nu}$, we have

$$nI(n) = (n-4)\int d^n l \frac{l^2}{(a+l^2)^3},$$

which we tackle using the Schwinger substitution as usual:

$$\begin{aligned} I(n) &= \frac{n-4}{n}\int d^n l \frac{l^2}{(a+l^2)^3} \\ &= \frac{n-4}{2n}\int_0^\infty d\alpha\, \alpha^2 \int d^n l\, l^2 \exp\left(-\alpha(a+l^2)\right). \end{aligned}$$

Convert to hyperspherical polar coordinates; $l^2 \to r^2$, $d^n l \to r^{n-1} dr d\Omega$, where $d\Omega$ the usual spherical area element.

$$\begin{aligned} &= \frac{n-4}{2n}\int d\Omega \int_0^\infty d\alpha\, \alpha^2 e^{-\alpha a}\int_0^\infty dr\, r^{n+1}\exp\left(-\alpha r^2\right) \\ &= \frac{n-4}{2n}\frac{n\pi^{n/2}}{\Gamma(1+n/2)}\int_0^\infty \frac{d\alpha}{\alpha}\alpha^3 e^{-\alpha a}\int_0^\infty \frac{dr}{r} r^{n+2}\exp\left(-\alpha r^2\right), \end{aligned}$$

where we have used the surface area of the general n-ball. Substitute $s = \alpha r^2$, $ds/s = 2dr/r$:

$$\begin{aligned} &= \frac{(n-4)\pi^{n/2}}{4\Gamma(1+n/2)}\int_0^\infty \frac{d\alpha}{\alpha}\alpha^3 e^{-\alpha a}\int_0^\infty \frac{ds}{s}(s/\alpha)^{1+n/2}\exp\left(-s\right) \\ &= \frac{(n-4)\pi^{n/2}}{4\Gamma(1+n/2)}\int_0^\infty \frac{d\alpha}{\alpha}e^{-\alpha a}\alpha^{2-n/2}\Gamma(1+n/2) \\ &= \frac{(n-4)\pi^{n/2}}{4a^{n/2-2}}\Gamma(2-n/2), \end{aligned}$$

where the last gamma function evaluation relies on $n < 4$. Putting $n = 4 - \varepsilon$, we find

$$I = -\varepsilon\frac{\pi^2}{4a^{\epsilon/2}}\Gamma(\varepsilon/2) = -\frac{\pi}{2} + O(\varepsilon) \tag{5.17}$$

near $n = 4$. Therefore the answer is finite when we use dimensional regularisation, rather than 0.

We also note at this point that from a purely calculational perspective, there are no Feynman parameters left in the integrals, so the remaining integrals just contribute factors of $1/2$.

5.4. **The amplitude calculated using dimensional regularisation.** When we apply (5.17) to $2\mathcal{M}_L^{(1)}$, we obtain

$$\frac{-e^2 g}{8\pi^2 M}\left[k_{2\mu}k_{1\nu} - g_{\mu\nu}(k_1 \cdot k_2)\right][2]. \tag{5.18}$$

This is the missing term in GWW's calculation. Similarly, applying DREG to the divergent term in $2\mathcal{M}_T^{(1)}$ gives

$$\frac{-e^2 g}{8\pi^2 M}\left[g_{\mu\nu}\right][-(n-1)]. \tag{5.19}$$

It is manifest that adding this term cancels with the problematic term (5.15), giving $0\mathcal{M}_g$. The final amplitude is therefore

$$\mathcal{M}_{\text{DREG}} = \frac{-e^2 g}{8\pi^2 M}\left[k_{2\mu}k_{1\nu} - g_{\mu\nu}(k_1 \cdot k_2)\right]\left[2 + 3\tau^{-1} + 3\left(2\tau^{-1} - \tau^{-2}\right) f(\tau)\right], \tag{5.20}$$

which agrees with previous calculations of this amplitude, taking into account different conventions for the overall scaling.

6. Discussion

6.1. **Which result is correct?** Since we have two different ways of dealing with integrals of the form (5.16), each of which gives a different answer, we have to ask which is actually correct. Firstly, we can ask if the answer is gauge-invariant. This is because the gauge we choose is an entirely non-physical calculational specification, so the (physical) result should have no dependence on it. This is perhaps less helpful in this case, since we started with two calculations in different gauges using different regularisations[6]

The calculation we have made above shows that the answer is the same in at least two different gauges when using dimensional regularisation. It can be further extended to the renormalisable R_ξ gauge, as shown in [8], which confirms the gauge-invariance of $\mathcal{M}_{\text{DREG}}$.[7]

However, GWW's contention in [5, 6] is that neither calculation is correct, because they believe that $(n-4)I_{\mu\nu}$ is not continuous in dimension, and should simply be set equal to 0 for $n = 4$. We shall demonstrate some problems with this in the next section. However, there is the more fundamental question of their treatment of the term $\mathcal{M}_g$. It is clear that this term should not be present in the answer, mainly because $\mathcal{M}_{\mu\nu}$ (this being more correct notation for the amplitude without the photons) must satisfy the Ward identities

$$\mathcal{M}_{\mu\nu}k_{1,2}^{\mu} = 0; \tag{6.1}$$

It is obvious that having tensor dependence only on $g_{\mu\nu}$ makes this impossible. Since the Ward identities are fundamentally related to gauge invariance (being a manifestation of the invariance of the functional measure under gauge transformations), $\mathcal{M}_g$ cannot be present for the answer to be properly gauge-invariant. Therefore GWW suggest using Dyson subtraction to remove the term. This is not actually valid for finite gauge invariance-violating terms (being a renormalisation technique), but this is not relevant to the current argument: we can show in other ways that the calculation is improper.

[6]Although GWW do not consider their calculation as containing a regularisation, we will use it as a shorthand for their result $I_{\mu\nu} = 0$.

[7]Obviously this calculation contains the complications of both multiterm propagators and the extra diagrams from non-unitary gauge, making it too complicated to investigate in detail here.

6.2. **The 't Hooft-Feynman gauge calculation.** We quote the result from [9], equation (11) for the usual calculation, but with the divergent integral unevaluated:

$$\begin{aligned} \mathcal{M}^{\mu\nu}_{\xi=1} = \frac{e^2 g}{(4\pi)^2 M} \Bigg[& -k_2^\mu k_1^\nu \left(2 + 3\tau + 3\tau\left(2-\tau\right) f\left(\tau\right)\right) \\ & - 2m_H^2 \left(1 + \frac{3}{2}\tau\right) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^4 l}{i\pi^2} \frac{g^{\mu\nu} l^2 - 4 l^\mu l^\nu}{\left(l^2 - 1 + 4x_1 x_2 \tau + i\epsilon\right)^3} \\ & + \frac{1}{2} m_H^2 g^{\mu\nu} \left(1 + \frac{3}{2}\tau + 3\tau\left(2-\tau\right) f\left(\tau\right)\right) \Bigg] \end{aligned} \tag{6.2}$$

We know that using dimensional regularisation gives the answer $\mathcal{M}^{\mu\nu}_{DREG}$ obtained above. Suppose, however, that the integral has some other value, i.e. we have

$$\mathcal{M}^{\mu\nu}_{\xi=1} = \frac{e^2 g}{(4\pi)^2 M} \left[-k_2^\mu k_1^\nu \left(2 + 3\tau + 3\tau\left(2-\tau\right) f\left(\tau\right)\right) + I g^{\mu\nu}. \right] \tag{6.3}$$

Now, we can assign a value to the potentially indeterminate integral in order to satisfy the Ward identity, *and this value is unique.* It is therefore unsurprising that the value obtained by the gauge-invariant DREG is the correct value, and it is impossible to consistently assign another value to the integral. For more on this idea, see Section 6.4 below.

However, in unitary gauge, recall that we have the integral with numerator given in (5.7), which [9] call A'. It contributes to both the $k_{2\mu} k_{1\nu}$ and $g_{\mu\nu}$ terms, which means it is no longer possible to fix the amplitude based on a Ward identity argument as we did above. As we saw above, this is where the missing term containing a 2 originates.

6.3. **Cutoff regularisation.** This section follows the ideas of [9]. We now look in more detail at $I_{\mu\nu}$, to determine if there is a consistent way to assign it a value in line with GWW's philosophy of finiteness and simplicity. We therefore consider a cutoff regularisation, and show why such a scheme is doomed to fail.[8] We start by considering the spherical cutoff implicitly used by GWW. For our purposes, we only need consider

$$\mathcal{I}_{\mu\nu} := \int d^4 l \frac{l^2 \delta_{\mu\nu} - 4 l_\mu l_\nu}{(1 + l^2)^3} \tag{6.4}$$

in Euclidean space. We will only look at $\mathcal{I}_{11}$, since the rest can be treated similarly:

$$\mathcal{I}_{11} := \int d^4 l \frac{l^2 - 4 l_1^2}{(1 + l^2)^3} \tag{6.5}$$

6.3.1. *Simple spherical cutoff.* We can make the spherical substitution

$$\begin{aligned} l &= r(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi\cos\psi, \sin\theta\sin\phi\sin\psi), \\ d^4 l &= r^3 \sin^2\theta \sin\psi \, dr d\theta d\phi d\psi, \end{aligned}$$

integrating to $r = \Lambda$, so the integral becomes

$$\begin{aligned} 4\pi \int_0^\Lambda dr \frac{r^5}{(1+r^2)^3} \int_0^\pi d\theta \sin^2\theta (1 - 4\cos^2\theta) \\ \propto \int_0^\pi d\theta (\cos 4\theta - \cos 2\theta) \\ = 0 \end{aligned}$$

So the integral disappears, as in GWW.

[8] Thus forcing us to regularise the integral in some way, or use the Ward identities argument of the previous section.

6.3.2. *Ellipses.* We next consider a different cutoff, biased towards the 1 direction: the ellipsoid

$$\frac{l_1^2}{1+\eta} + l_2^2 + l_3^2 + l_4^2 \leq \Lambda^2;$$

which is simpler to work with by changing variables to

$$\begin{aligned} l &= r\big(\sqrt{1+\eta}\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi\cos\psi, \sin\theta\sin\phi\sin\psi\big), \\ d^4 l &= \sqrt{1+\eta} r^3 \sin^2\theta \sin\psi\, dr d\theta d\phi d\psi. \end{aligned}$$

The integral now becomes

$$\begin{aligned} \mathcal{I}_{11} &= 4\pi\sqrt{1+\eta}\int_0^\Lambda dr r^5 \int_0^\pi d\theta \sin^2\theta \frac{1-(4+3\eta)\cos^2\theta}{(1+r^2+r^2\eta\cos^2\theta)^3} \\ &= \frac{\pi^2\sqrt{1+\eta}\Big(8+(8-(-4+\eta)\eta)\Lambda^4 - 8\sqrt{(1+\Lambda^2)(1+(1+\eta)\Lambda^2)} + 4\Lambda^2\Big(4+\eta-2\sqrt{(1+\Lambda^2)(1+(1+\eta)\Lambda^2)}\Big)\Big)}{2\eta^2(1+\Lambda^2)^{3/2}\sqrt{1+(1+\eta)\Lambda^2}} \\ &= \frac{\pi^2\left(8+4\eta-\eta^2-8\sqrt{1+\eta}\right)}{2\eta^2} + O(\Lambda^{-2}), \end{aligned}$$

which clearly depends explicitly on the cutoff geometry. The situation is even worse than it first appears: considering the other components of $\mathcal{I}$ with the same cutoff, we notice that, for example,

$$\begin{aligned} \mathcal{I}_{22} &= 2\pi\sqrt{1+\eta}\int_0^\Lambda dr r^5 \int_0^\pi d\theta \sin^2\theta \int_0^\pi d\phi \sin\phi \frac{1+\eta\cos^2\theta - 4\cos^2\phi\sin^2\theta}{(1+r^2+r^2\eta\cos^2\theta)^3} \\ &= \frac{4\pi}{3}\sqrt{1+\eta}\int_0^\Lambda dr r^5 \int_0^\pi d\theta \sin^2\theta \frac{-1+(4+3\eta)\cos^2\theta}{(1+r^2+r^2\eta\cos^2\theta)^3} \\ &= -\frac{1}{3}\mathcal{I}_{11}, \end{aligned}$$

and we have even lost tensor invariance! Clearly using a momentum cutoff on this integral is not only dangerous for gauge-invariance,[9] but also, using this technique the integral also undetermined in a non-trivial fashion. In the next section we explain the basic general reason that this is so. [9] gives some more examples of similar integrals with this property. The "symmetric integration" of GWW is simply unjustifiable.

6.3.3. *Riemann's theorem on series summation.* For large r, the integral over the section of the sphere where the integrand is positive can be seen to be of order $1/r$.[10] It therefore seems appropriate to consider the series

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots,$$

which is of the same order of magnitude as the integration if we cut both off at Λ. The above series clearly converges to 0, because the successive sums are

$$1, 0, \frac{1}{2}, 0, \frac{1}{3}, \ldots$$

However, the well-known theorem of Riemann concerning series rearrangement[11] shows that we can make this series have any real number we choose as a sum, because the sum of the absolute values of the coefficients is divergent (being effectively the harmonic series). If the positive terms are a_i and the negative $-b_i$, to reach the value α, we sum the a_i until the total is larger than $\alpha + b_1$, then subtract until the value is less than α minus the next a_i, etc. Choosing the contour to evaluate the integral is exactly the same idea.

We therefore conclude that as an integral in exactly 4 dimensions, $\mathcal{I}_{\mu\nu}$ is impossible to include in the algebra as a meaningful quantity. GWW's treatment of it should therefore be considered erroneous, and we *must* use some sort of regularisation to obtain the correctly gauge-invariant (and indeed, physically sound) amplitude.

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[9] Which we expect for cutoff regularisation anyway; see [12, 13].

[10] Precisely, it is $3\sqrt{3}\pi/2r + O(r^{-3})$.

[11] [10], pp.8-9

Nevertheless, we feel that GWW's criticism of dimensional regularisation has perhaps not been answered by the above discussion, so we shall end the paper by discussing a new, alternative method of regularisation, which is usable in 4 dimensions directly, yet still treats the divergent integrals correctly.

6.4. **Another regularisation scheme, LORE.**

6.4.1. *The theory.* In [12, 13], a new regularisation technique is suggested, with the key characteristics that

(1) The calculation takes place in 4 dimensions throughout.
(2) The diagram integrals are the pieces of the calculation regularised, not the Lagrangian or the propagators themselves.
(3) There are introduced 2 mass scales, μ_s and M_c, which control the IR and UV divergence respectively. These are to be compared to the bald cutoff Λ used in cutoff regularisation. The contrast is that the former maintain both Lorentz and gauge invariance, which are lost by the latter.

However, we do not need the full power of this formalism for the simple loop integrals we have here. We instead use the ideas of section II of the paper, similarly to [7]. We define the irreducible loop integrals (ILIs) we need as follows:

$$I_{2\alpha}(m) = \int d^4 l \frac{1}{[l^2 - m^2]^\alpha}, \tag{6.6}$$

$$I_{2\alpha\mu\nu}(m) = \int d^4 l \frac{l_\mu l_\nu}{[l^2 - m^2]^{\alpha+1}}, \tag{6.7}$$

where the subscript numbers indicate the asymptotic degree of the integrand, so I_0, for example, is logarithmically divergent, and I_{-2} converges, and has the value[12]

$$\begin{aligned} I_{-2} &= \int d^4 l \frac{1}{[l^2 - M^2 + 2\alpha_1\alpha_2(k_1 \cdot k_2)]^3} \\ &= -\frac{i\pi^2}{2}\frac{1}{m^2} \end{aligned} \tag{6.8}$$

In the subsequent discussion we shall refer to the case $m^2 = M^2 - 2\alpha_1\alpha_2(k_1 \cdot k_2)$ by dropping the m dependence in the $I_{2\alpha}(m)$ notation.[13]

There is derived in section II of [12, 13] a relationship between the scalar and tensor integrals of the same degree. We shall explain the process for the quadratically divergent ILIs; the process is similar for the logarithmically divergent ones, but because it uses three diagrams, the algebra is much more complicated.

The general Lagrangian of Yang-Mills theory is written down, then the four different vacuum diagrams are computed. The fermionic diagram is shown to satisfy the Ward identities separately from the others; the diagram is easily shown to have amplitude proportional to

$$\Pi^{(f)}_{\mu\nu} = \int_0^1 dx \left[2I_{2\mu\nu}(m) - I_2(m)g_{\mu\nu} + 2x(1-x)(p^2 g_{\mu\nu} - p_\mu p_\nu)I_0(m)\right], \tag{6.9}$$

where m has a dependence on x.

To impose a gauge invariant regularisation, we need $p^\mu \Pi^{(f)}_{\mu\nu} = 0$, the Ward identity in question. We see immediately that the quadratically divergent ILIs have to be removed, since they contain no p dependence; to do this, our regularisation must impose the condition

$$I^R_{2\mu\nu} = \frac{1}{2} I^R_2 g_{\mu\nu},$$

[12]We have in fact calculated this integral above in Section 5.2.

[13]As we did when discussing the finite I_{-2} in Section 5.2

where I^R denotes the regularised value of the integral I. This condition must be imposed, *whatever regularisation we use*. Looking at the non-fermionic vacuum diagrams in a similar way, after a lot of algebra, we find the consistency condition we actually want,

$$I^R_{0\mu\nu} = \frac{1}{4} I^R_0 g_{\mu\nu}. \tag{6.10}$$

Therefore, we can obtain a result without actually using a specific regularisation: this condition is an entirely general necessary condition for gauge invariance of the vacuum amplitude of any Yang-Mills theory.[14]

6.4.2. *Application to* $\mathcal{M}$. We once again consider $\mathcal{M}^{(1)}_L$ and $\mathcal{M}^{(1)}_T$, which we rearrange into the new forms in [7]

$$\begin{aligned} \mathcal{M}^{(1)}_L = \frac{-4ie^2 g}{(2\pi)^4 M} \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \\ \Big(4\left[(k_1 \cdot k_2) g^\rho_\mu g^\sigma_\nu - k^\sigma_2 k_{1\nu} g^\rho_\mu - k^\rho_1 k_{2\mu} g^\sigma_\nu + k^\rho_1 k^\sigma_2 g_{\mu\nu}\right] I_{0\mu\nu} \\ + 2\left[k_{2\mu} k_{1\nu} - g_{\mu\nu}(k_1 \cdot k_2)\right] I_0 \\ + 2\left[k_{2\mu} k_{1\nu} - g_{\mu\nu}(k_1 \cdot k_2)\right]\left[M^2 - 2\alpha_1\alpha_2(k_1 \cdot k_2)\right] I_{-2}\Big), \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} \mathcal{M}^{(1)}_T = \frac{-4ie^2 g}{(2\pi)^4 M} \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \\ \Big((-3)\left(g_{\mu\nu} I_0 - 4 I_{0\mu\nu}\right) \\ + 6\left[k_{2\mu} k_{1\nu} - g_{\mu\nu}(k_1 \cdot k_2)\right](1 - 2\alpha_1\alpha_2) I_{-2}\Big). \end{aligned} \tag{6.12}$$

The important difference is that in both, the manipulation of the fractions there has caused to appear a new instance of the finite integral I_{-2}: this time it is this that cancels the non-gauge-invariant term in $\mathcal{M}^{(1)}_T$, and is the whole contribution from $\mathcal{M}^{(1)}_L$. It is obvious that our conditions for gauge invariance cause all the divergent terms to cancel, leaving us once again with the answers

$$\mathcal{M}^{(1)}_L = -\frac{-e^2 g}{16\pi^2 M} 2\left[k_{2\mu} k_{1\nu} - g_{\mu\nu}(k_1 \cdot k_2)\right] \tag{6.13}$$

$$\mathcal{M}^{(1)}_T = \frac{-e^2 g}{16\pi^2 M}\left[k_{2\mu} k_{1\nu} - g_{\mu\nu}(k_1 \cdot k_2)\right]\left[3\tau^{-1} + 3\left(2\tau^{-1} - \tau^{-2}\right) f(\tau)\right]. \tag{6.14}$$

It is clear that these once again produce $\mathcal{M}_{\mathrm{DREG}}$ when we add them together with the $\mathcal{M}^{(3)}$ terms. We have not used loop regularisation explicitly: we only needed the technique's existence.

7. Conclusion

Using dimensional regularisation in unitary gauge, we have obtained the standard result. This shows that the use of unitary gauge is not inherently the problem, although the algebra is unpleasant.

The result of [5, 6] is incorrect; it is incorrect because when striving for a finite calculation, they did not treat the divergent integrals with any regularisation, which we showed was necessary for consistency and gauge invariance in section 6.4. This problem is not noticeable in unitary gauge because the integral can contribute to both tensorial terms in the amplitude, which disallows us

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[14] It is worth remarking that we seem to have assumed that such a regularisation exists. However, we are fortunate here: our calculation in Section 5.3 showed that dimensional regularisation gives precisely the correct relationship for $I_{0\mu\nu}$ and I_0.[15]

[15] It is not as if physics has not proceeded without the existence of a mathematical entity before: both the lack of a translation-invariant measure on spaces used in functional integration and the lack of an interaction picture in QFT have not stood in the way of progress.

from being able to apply a "tuned" value to the integral *a posteriori* to satisfy gauge invariance. We also investigated the calculation in 4 dimensions directly with the loop regularisation technique and found the answer agreed with the standard result $\mathcal{M}_{\text{DREG}}$.

Other papers have examined the calculation using unitary gauge and dimensional regularisation, R_ξ gauge and dimensional regularisation [8], unitary gauge and loop regularisation [7], and 't Hooft-Feynman gauge with cutoff regularisation [11]. We have tied several of these results together in the preceding section. There has also been executed a numerical calculation using a lattice regulator that agrees to high accuracy with the usual DREG result [1].

We hope that the extensive discussion that has been aroused by this calculation has contributed to the mathematical understanding of the mechanisms involved, and the resolution proves useful in future.

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