Internship Paper, July 2012

Richard Chapling

August 18, 2012

1 Summary

I was fortunate to be given the opportunity to take up an internship in July 2012.

During the course of this internship, I was able to explore in depth the work of Professor Robert Woodhouse (1773-1827) and his work in attempting to introduce the continental calculus notation to Cambridge (i.e. England), his attitude to "analysis" and algebraic reasoning and the extent of application of algebraic techniques. Much of the literature on Woodhouse has failed to interpret the context of his position in Cambridge:[1] there is an enlightening paper by Christopher Phillips on the subject, [9], which inspired some of the remarks made below.

The significance of Woodhouse's work is as the first attempt to introduce the continental notation for the calculus to England. It has long been a prevailing view that the Newtonian fluxional notation was a chief reason for the slow progression of the calculational science in England in the century after Newton. While this is far from the whole picture, Woodhouse felt strongly that the Newtonian notation was inferior in expression to the continental: he says that "[...] its advantages ought [...] to be great and manifest, since they have caused me to act in opposition to early habits, to the feelings of national prejudice, to my more than rational reverence of Newton".

I was also able to prove several conjectures concerning certain sums that are important theoretically in the physical technique of loop regularisation (LORE). A draft of the paper is enclosed, and a short note on the background is given below in Section 3.

2 Woodhouse and the Tripos

Woodhouse's first book was the *Principles of Analytical Calculation*, [16]. (It is necessary at this point to mention that the word "analysis", in Woodhouse's day, essentially meant Not Synthetic Geometry; it could be algebraic manipulation, or what we today call calculus, or half-a-dozen other mathematical areas.) This was mainly a large treatise on the calculus, and its most significant feature is the use of differential notation: it uses both d/dx and D, and no fluxional dots. Woodhouse gives two reasons for this choice: firstly he believes that it looks clearer. More significant from a philosophical point of view is his discussion in the Preface of what he sees as the inadequate basis for the fluxional calculus, derived from motion; he wishes to remedy this (and, hence, inter alia, to some extent, answer Bishop Berkeley, for whom he claims much respect) by following Lagrange's programme of "Algebraic fluxional analysis". While he approves of Lagrange's ideas, he is critical of his exposition (this is perhaps somewhat hypocritical, given that Rouse Ball would describe the *Principles* as difficult to read due to the "extra-ordinary complication of grammatical construction in which he revels"[2]); of course, this criticism had the convenient ulterior objective of enabling the text to appear more conservative than it actually was: in its presence, Woodhouse could hardly be accused of being simply a radical mouthpiece for Dangerous Foreign Ideas; he instead presents the differential notation and Algebraic approach as one of several different ways of deriving and

[1] [4] is a fairly typical example, although the author does make a number of sensible comments in the direction of our thesis.

[2] [10], p 118

utilising the fluxional calculus, but he believes it is (as mentioned above) the clearest route to the conclusions reached.

The *Principles* had little influence on those who were not members of the Analytical Society; this probably has its origins in that Great Cambridge Institution, the Tripos: very little original mathematics was actually done at Cambridge in this period, since the University was mainly concerned with teaching undergraduates to pass their Exams, so they could leave and become Proper Gentlemen; as such, the most important topic in Mathematics at Cambridge was What Was On The Tripos, and if a book was to have any influence, it needed to both contain Mathematics That Could Be Set For Undergraduates (uncontroversial, "complete" topics that had simple exposition and rigorous foundation, for a value of rigour acceptable to both the Conservative Liberals and the Liberal Conservatives in the Senate), and contain Problems on which the undergraduates could be Tutored (while this was not yet the era when passing Tripos required a Tutor, Brains, Endurance, and little else, it was not far off, and the majority of undergraduate education was from tutoring); Woodhouse appears to have written his later texts with a view to this situation, as we shall see below.

Woodhouse's second published work using the differential notation was his 1810 book, *A Treatise on Isoperimetrical Problems and the Calculus of Variations*, [12]. Once again, the Preface is significant: Woodhouse explains that he wishes to write an up-to-date English text on a subject in which there are few expository texts, the most significant being by Euler; his intention is to describe the subject's historical development, because he believes that "the student's curiosity will be more excited and sustained, when he finds history blended with science, and the demonstration of formulae accompanied with the object and causes of their invention."; this is therefore the first of many calculus textbooks (for it will be shewn that this is a textbook, id est, a book for undergraduate use in preparing for Tripos) that uses historical discussion in its exposition. Woodhouse's justification for differential notation this time is that it allows him to use δ as analogous to d for variational operations, following Lagrange, and maintaining his critical stance towards the methods of foreigners, he declares that Lagrange has "power over symbols [...] so unbounded that the possession of it seems to have made him capricious", and goes on to speak of the "plain and palpable evils of a confusion in the signification of symbols". He also discusses the diminishing use of diagrams over the course of the subject's history, and declares that "A similar history will belong to every other method of calculation, that has been advanced to any degree of perfection.". The last 40 pages contain numerous worked problems, eminently suitable for Tripos study, which Peacock would indeed take advantage of in subsequent years.

A year earlier, Woodhouse had published *A Treatise on Plane and Spherical Trigonometry*, [13], where he started with the algebraical definition of the sine and cosine and derived the rest with no geometry, although it still contained some diagrams; it also contained numerous worked examples; Peacock said that it, "more than any other work contributed to revolutionize the mathematical studies of this country".[3] It was also this work that Peacock described as "opposed and stigmatized by many of the older members, as tending to produce a dangerous innovation in the existing course of academical studies, and to subvert the prevalent taste of the geometrical form of conducting investigations";[4] nevertheless, the *Trigonometry* went through four editions in the decade following its publication, and was so thoroughly regarded as Useful For Tripos that Airy was studying it even before he came up.[5]

At this point the Analytical Society came into play: many of its members (Babbage, Peacock) citing Woodhouse's Principles for inspiring their interest in the differential notation. Woodhouse's influence did not end there: the single volume of its Memoirs also contains a trigonometrical investigation parallel to Woodhouse's, but as a new piece of advanced original work, rather than Woodhouse's intention to rederive the known results with Analysis.[6] The infamous translation of Lacroix published in 1816, [5], was even less studied than the *Principles*, but Peacock had begun to realise the folly of the introduction of such a text to Cambridge, especially with Babbage

[3] [8], p. 295
[4] *ibid.*, p. 296
[5] [1], p. 20
[6] [11]

and Herschel's attempts to introduce even more advanced material, and his influence lead to the publication of *A Collection of Examples of the Applications of the Differential and Integral Calculus* in 1820, [6], which were much more suitable for study (the Woodhouse influence in these is considerable: Woodhouse had written in his early papers and the *Principles* discussing the nature of Equality; in particular, the difference between arithmetical and algebraical equality, which would influence Peacock enormously in his subsequent work, right down to the publication of his *Treatise on Algebra* in 1830, [7]: it is here that the Principle of Permanence of Forms as the mediator is first posited, as the answer to bridging the gap between the two[7]).

Meanwhile, Woodhouse had moved on to write about astronomy; it is easy to imagine why once we consider the view of astronomy in Cambridge at this time: it was essentially a complete science, initiated in its current form by Newton, and developed into the highest and most complex application of trigonometry, and Woodhouse set out to transform the subject to an application of analysis. *An Elementary Treatise on Astronomy* (1812, [15]) was wildly successful from the point of view of reforming the curriculum: it, together with the *Calculus of Variations* and the *Trigonometry*, were cited over 120 times as Useful For Tripos before 1820, and Peacock would set over 10 questions in their style in 1817; it was even said in the *Edinburgh Review* in a review of his next book that "No man has done so much to improve the studies of Cambridge as Mr Woodhouse.".[8] As a description of the experimental end it was also highly praised: De Morgan said in his *Penny Cyclopaedia* article on Woodhouse that "the examples seem as if they were real ones, as if some astronomer had had to put down the actual figures, and the very observations which are cited are made to smell of the instruments which gave them"

After the publication of his *Physical Astronomy*, [14], the sequel to the *Elementary Astronomy*, and mainly concerned with Newtonian gravity and the three-body problem, Woodhouse's influence essentially ended: opposition by Whewell and his successor Airy to his analytical viewpoint would cause them to much to diffuse Woodhouse's reputation and the influence of his work: their Tripos questions made Woodhouse's texts of little use to undergraduates once again. Airy was nothing if not efficient in his displacement of Woodhouse: he had assessed the value of Woodhouse's estate and furniture prior to his election to the Plumian Professorship after Woodhouse's death, and auctioned off the furniture less than a week after taking office, and attempted to have Woodhouse's estate pay for the repairs to the house.[9] Woodhouse's papers in the house went missing without trace, explaining much of our lack of knowledge of Woodhouse.

The prior convention has been to assign Woodhouse the rôle of one who attempted to introduce differential notation, and failed; so only the younger generation of the Analytical Society could introduce reform; however, in the context of the situation of Cambridge under the old Tripos and the old Newtonianism, one can view Woodhouse's overall career goal of introducing Analysis to Cambridge as a success, since his later books became the basis of numerous Tripos questions, and brought about the utter transformation of both notation and curriculum in his own lifetime. Our knowledge of Woodhouse is severely curtained by Airy, as he took over the lead in Astronomy and gave Woodhouse essentially no credit for his influence.[10]

3 The Asymptotics of Certain Sums Containing Binomial Coefficients

While writing my Part III essay related to calculating the decay rate of the Higgs boson into two photons, I read a paper [17] which discussed the technique of loop regularisation (material from which enabled me to demonstrate the correctness of a certain calculation without having to appeal to a particular method of regularisation). In this paper were three conjectures (their (4.5), (4.7), (4.9)) relating to the asymptotic behaviour of certain sums; these are used to obtain

[7]See [3] for a fuller discussion of Woodhouse's algebraic ideas in the *Principles*

[8][2], p. 394

[9]These are taken from Letters 46-8 of Airy's papers in the Wren Library.

[10]See [9] for more detail on this and other topics discussed here.

a considerable simplification of the expressions involved in the subsequent discussion, and have profound importance in the theoretical underpinnings of its calculations.

While I was working on an unrelated problem, I discovered a set of integrals,

$$I(m, n, s) := \int_0^\infty y^{n-1}(1 - e^{-y})^m e^{-sy}\, dy,$$

that, for certain values of the parameters, when evaluated exactly, will give the complicated sums in the conjectures; I was then able to obtain the conjectured asymptotic expansions by approximating the same integral. In two cases, this requires subtracting terms to regulate the integral in the limit as $s \to 0$; this was necessary to maintain the simplicity of the rest of the analysis.

References

[1] G.B. Airy and W. Airy. *Autobiography of Sir George Biddell Airy.* Cambridge University Press, 1896.

[2] Anonymous. "Review of Woodhouse's Physical Astronomy". In: *Edinburgh Review* xxxi (1819), pp. 375–94.

[3] Harvey W Becher. "Woodhouse, Peacock, Babbage and Modern Algebra". In: *Historia Mathematica* vii (1980), pp. 389–400.

[4] W Johnson. "Contributors to Improving the Teaching of Calculus in Early 19th-Century England". In: *Notes and Records of the Royal Society of London* 49.1 (1995), pp. 93–103.

[5] Silvestre François Lacroix et al. *An Elementary Treatise on the Differential and Integral Calculus.* Printed by J. Smith for J. Deighton and sons, 1816.

[6] G. Peacock. *A Collection of Examples of the Applications of the Differential and Integral Calculus.* Printed by J. Smith, sold by J. Deighton and sons, 1820.

[7] George Peacock. *A Treatise on Algebra.* Cambridge University Press, 1830.

[8] George Peacock. "Report on the Recent Progress and Present State of certain Branches of Analysis". In: *Report of the Third Meeting of the British Association for the Advancement of Science* (1834), pp. 185–352.

[9] C. Phillips. "Robert Woodhouse and the Evolution of Cambridge Mathematics". In: *History of Science* 44 (2006), pp. 69–93.

[10] W W Rouse Ball. *A History of the Study of Mathematics at Cambridge.* Cambridge University Press, 1889.

[11] Analytical Society. "On Trigonometrical Series; Particularly those whose Terms are multiplied by the Tangents, Cotangents, Secants, &c of Quantities in Arithmetic Progression; together with some singular Transformations." In: *Memoirs of the Analytical Society* (1813), pp. 33–46.

[12] Robert Woodhouse. *A Treatise on Isoperimetrical Problems, and the Calculus of Variations.* Cambridge University Press, 1810.

[13] Robert Woodhouse. *A Treatise on Plane and Spherical Trigonometry.* J. Deighton, 1809.

[14] Robert Woodhouse. *An Elementary Treatise on Astronomy Vol. II, Containing Physical Astronomy.* J. Smith, Printer to the University, 1818.

[15] Robert Woodhouse. *An Elementary Treatise on Astronomy.* J. Smith, Printer to the University, 1812.

[16] Robert Woodhouse. *The Principles of Analytical Calculation.* Cambridge University Press, 1803.

[17] Yue-Liang Wu. *Symmetry Principle Preserving and Infinity Free Regularization and renormalization of quantum field theories and the mass gap.* Nov. 2008. URL: http://arxiv.org/abs/hep-th/0209021v4.