Summary of Convergence Tests

Richard Chapling

v1 12 February 2019

This sheet covers results about convergence of series, and all the common tests. It does not include any results about the *value* of a convergent series, apart from the elementary ones about geometric and telescoping ones. (We have also given no results about uniform convergence or Fourier series, which have their own theory, only covered in later courses.)

0.0.1 Notation

Unless otherwise stated, $\sum_{n=0}^{\infty} a_n$ is the series we want to test. Appended to a series, (C) means convergent, $|C|$ absolutely convergent, and (D) divergent; by themselves they mean that $\sum_{n=0}^{\infty} a_n$ is convergent, i.e.

$$(C) \equiv \left(\sum_{n=0}^{\infty} a_n \quad (C) \right), \qquad \&c.$$

0.1 Basic results

0.1.1 Convergent implies partial sums bounded

$$(C) \implies \exists M \in [0,\infty) : \forall k \in \mathbb{N}, \left| \sum_{n=0}^{k} a_n \right| < M$$

0.1.2 Convergent iff partial sums Cauchy

$$(C) \iff \forall \varepsilon > 0, \exists N \in \mathbb{N} : (\forall k, m > N), \left| \sum_{n=k}^{m} a_n \right| < \varepsilon$$

0.1.3 Changing or omitting finitely many terms does not affect convergence

$$\sum_{n=0}^{\infty} a_n \quad (C) \implies \forall N \in \mathbb{N}, \sum_{n=N}^{\infty} a_n \quad (C)$$

0.1.4 Absolutely convergent implies convergent

$$|C| \implies (C)$$

0.1.5 Absolutely convergent implies any subseries converges

$$|C| \implies \sum_{k=0}^{\infty} a_{n(k)} \quad (C)$$

0.1.6 Absolutely convergent implies any rearrangement converges

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be injective. Then

$$|C| \implies \sum_{n=0}^{\infty} a_{\sigma(k)} \quad (C)$$

(And if σ is bijective, the sums are equal)

0.1.7 Term Test

$$a_n \nrightarrow 0 \implies (D)$$

0.2 Series we know about

0.2.1 Geometric series

$$\sum_{n=0}^{k} ar^n = \frac{a(1-r^{k+1})}{1-r}$$

If $-1 < r < 1$:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

0.2.2 Telescoping series

$$\sum_{n=0}^{k} (f(n) - f(n+1)) = f(0) - f(k+1)$$

If $f(k) \to 0$ as $k \to \infty$,

$$\sum_{n=0}^{\infty} (f(n) - f(n+1)) = f(0)$$

1 For Series with Positive Terms

We assume for all these that $a_n > 0$. Certain results also apply if $a_n \geqslant 0$; these are marked with ^{0}s.

1.1 General principles

If $a_n \geqslant 0$,

1.1.1 Convergent iff partial sums are bounded

$$\sum_{n=0}^{\infty} a_n \quad (C) \iff \exists M \in [0,\infty) : \forall k \in \mathbb{N}, \sum_{n=0}^{k} a_n < M$$

1.1.2 Divergent subseries implies divergent

$$\sum_{k=0}^{\infty} a_{n(k)} \quad (D) \implies \sum_{n=0}^{\infty} a_n \quad (D)$$

(Hence all divergence tests in this section need only be applied to a subseries to show divergence.)

1.2 Comparison Tests

1.2.1 Direct^0

$$\left. \begin{array}{l} a_n \leqslant c_n \\ \sum_{n=0}^{\infty} c_n \quad (C) \end{array} \right\} \implies \sum_{n=0}^{\infty} a_n \quad (C)$$

$$\left. \begin{array}{l} a_n \geqslant d_n \\ \sum_{n=0}^{\infty} d_n \quad (D) \end{array} \right\} \implies \sum_{n=0}^{\infty} a_n \quad (D)$$

1.2.2 Limit^0

$$\left. \begin{array}{l} \liminf_{n\to\infty} \frac{a_n}{c_n} > 0 \\ \sum_{n=0}^{\infty} c_n \quad (C) \end{array} \right\} \implies \sum_{n=0}^{\infty} a_n \quad (C)$$

$$\left. \begin{array}{l} \limsup_{n\to\infty} \frac{a_n}{d_n} < \infty \\ \sum_{n=0}^{\infty} d_n \quad (D) \end{array} \right\} \implies \sum_{n=0}^{\infty} a_n \quad (D)$$

1.2.3 Ratio

$$\left. \begin{array}{l} \frac{a_{n+1}}{a_n} \leqslant \frac{c_{n+1}}{c_n} \\ \sum_{n=0}^{\infty} c_n \quad (C) \end{array} \right\} \implies \sum_{n=0}^{\infty} a_n \quad (C)$$

$$\left. \begin{array}{l} \frac{a_{n+1}}{a_n} \geqslant \frac{d_{n+1}}{d_n} \\ \sum_{n=0}^{\infty} d_n \quad (D) \end{array} \right\} \implies \sum_{n=0}^{\infty} a_n \quad (D)$$

1.3 Ratio Tests

1.3.1 D'Alembert

Limit form

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \implies (C)$$

$$\exists N \in \mathbb{N} : \inf_{n>N} \frac{a_{n+1}}{a_n} \geqslant 1 \implies (D)$$

Non-limit form

$$\exists N \in \mathbb{N}, p < 1 : \forall n > N, \frac{a_{n+1}}{a_n} < p \implies (C)$$

$$\exists N \in \mathbb{N} : \forall n > N, \frac{a_{n+1}}{a_n} \geqslant 1 \implies (D)$$

1.3.2 Raabe

Limit form

$$\limsup_{n\to\infty} n \left(\frac{a_{n+1}}{a_n} - 1 \right) < -1 \implies (C)$$

$$\exists N \in \mathbb{N} : \inf_{n>N} n \left(\frac{a_{n+1}}{a_n} - 1 \right) \geqslant -1 \implies (D)$$

Non-limit form

$$\exists N \in \mathbb{N}, p < -1 : \forall n > N, \frac{a_{n+1}}{a_n} < 1 + \frac{p}{n} \implies (C)$$

$$\exists N \in \mathbb{N} : \forall n > N, \frac{a_{n+1}}{a_n} \geqslant 1 - \frac{1}{n} \implies (D)$$

1.3.3 Bertrand

Limit form

$$\limsup_{n\to\infty} \log n \left(n \left(\frac{a_{n+1}}{a_n} - 1 \right) + 1 \right) < -1 \implies (C)$$

$$\exists N \in \mathbb{N} : \inf_{n>N} \log n \left(n \left(\frac{a_{n+1}}{a_n} - 1 \right) + 1 \right) \geqslant -1 \implies (D)$$

Non-limit form

$$\exists N \in \mathbb{N}, p < -1 : \forall n > N, \frac{a_{n+1}}{a_n} > 1 - \frac{1}{n} + \frac{p}{n \log n} \implies (C)$$

$$\exists N \in \mathbb{N} : \forall n > N, \frac{a_{n+1}}{a_n} \geqslant 1 - \frac{1}{n} - \frac{1}{n \log n} \implies (D)$$

1.3.4 Gauss

If we have

$$\frac{a_{n+1}}{a_n} = 1 + \frac{p}{n} + o\left(\frac{1}{n^{1+\delta}}\right)$$

for some $\delta > 0$, then

$$p < -1 \implies (C)$$
$$p \geqslant -1 \implies (D).$$

1.3.5 Kummer

If b_n is any nonnegative sequence:

$$\limsup_{n\to\infty} b_{n+1}\frac{a_{n+1}}{a_n} - b_n < 0 \implies (C)$$

If d_n is any nonnegative sequence with $\sum_{n=0}^{\infty} d_n$ (D):

$$\exists N \in \mathbb{N} : \inf_{n>N} d_{n+1}\frac{a_{n+1}}{a_n} - d_n \geqslant 0 \implies (D)$$

1.3.6 Schlömilch–Páras

Let

$$B(r,x) = \begin{cases} \frac{x^r-1}{r} & r \neq 0 \\ \log r & r = 0 \end{cases}.$$

Then

$$\exists r \in \mathbb{R} : \limsup_{n\to\infty} B(r, a_{n+1}/a_n) < -1 \implies (C)$$
$$\exists r \in \mathbb{R}, N \in \mathbb{N} : \inf_{n>N} B(r, a_{n+1}/a_n) \geqslant -1 \implies (D)$$

1.3.7 Second and mth Ratio Tests

Second ratio test:

$$\max\left\{\limsup_{n\to\infty}\frac{a_{2n}}{a_n}, \limsup_{n\to\infty}\frac{a_{2n+1}}{a_n}\right\} < \frac{1}{2} \implies (C)$$
$$\exists N \in \mathbb{N} : \min\left\{\inf_{n>N}\frac{a_{2n}}{a_n}, \inf_{n>N}\frac{a_{2n+1}}{a_n}\right\} \geqslant \frac{1}{2} \implies (D)$$

mth ratio test:

$$\max_{0\leqslant k\leqslant m-1}\left\{\limsup_{n\to\infty}\frac{a_{mn+k}}{a_n},\right\} < \frac{1}{m} \implies (C)$$
$$\exists N \in \mathbb{N} : \min_{0\leqslant k\leqslant m-1}\left\{\inf_{n>N}\frac{a_{mn+k}}{a_n}\right\} \geqslant \frac{1}{m} \implies (D)$$

1.4 Root Tests

1.4.1 Cauchy[0]

$$\limsup_{n\to\infty} a_n^{1/n} \begin{cases} < 1 & (C) \\ > 1 & (D) \end{cases}$$

1.4.2 Logarithmic[0]

$$\exists m : \limsup_{n\to\infty} \frac{\log a_n + \log n + \cdots + \log^{\circ m} n}{\log^{\circ m} n} < 0 \implies (C)$$

$$\exists m, N : \inf_{n>N} \frac{\log a_n + \log n + \cdots + \log^{\circ m} n}{\log^{\circ m} n} \geqslant 0 \implies (D)$$

1.5 For Series with Nonincreasing Positive Terms

1.5.1 Cauchy Condensation

$$\sum_{n=0}^{\infty} a_n \quad (C) \iff \sum_{k=0}^{\infty} 2^k a_{2^k} \quad (C)$$

1.5.2 Schlömilch Condensation

Let $u\colon \mathbb{N} \to [0,\infty)$ be an increasing, unbounded function, and there is $c > 0$ so that

$$\frac{u(n+1)-u(n)}{u(n)-u(n-1)} \leqslant c.$$

$$\sum_{n=0}^{\infty} a_n \quad (C) \iff \sum_{k=0}^{\infty} \big(u(n+1)-u(n)\big)a_{u(n)} \quad (C)$$

1.5.3 Lobachevsky's Test

Let $p(k)$ be the largest index so that $a_{p(k)} \geqslant 2^{-k}$.

$$\sum_{n=0}^{\infty} a_n \quad (C) \iff \sum_{k=0}^{\infty} 2^{-k} p(k) \quad (C)$$

A generalisation: let (b_k) be positive with

$$0 < \alpha \leqslant \frac{b_{k+1}}{b_k} \leqslant \beta < 1$$

for all k, and now let $p(k)$ be the largest index so that $a_{p(k)} \geqslant \sum_{n=k}^{\infty} b_n$.

$$\sum_{n=0}^{\infty} a_n \quad (C) \iff \sum_{k=0}^{\infty} p(k) b_k \quad (C)$$

1.5.4 Maclaurin's Integral Test

Let $f\colon [0,\infty) \to [0,\infty)$ be nonincreasing and $f(x) \to 0$ as $x \to \infty$. Then

$$\lim_{N\to\infty} \sum_{n=0}^{N} f(n) - \int_0^{N+1} f$$

is finite, and

$$\sum_{n=0}^{\infty} f(n) \quad (C) \iff \int_0^{\infty} f \quad (C)$$

1.5.5 De la Vallée Poussin's Integral Test

Let $(A_n)_{n=0}^{\infty}$ be increasing and unbounded, $f\colon [0,\infty) \to [0,\infty)$ be nonincreasing with $f(x) \to 0$ as $x \to \infty$. Then

$$\int_0^{\infty} f \quad (C) \implies \sum_{n=0}^{\infty} (A_{n+1}-A_n) f(A_{n+1}) \quad (C)$$
$$\int_0^{\infty} f \quad (D) \implies \sum_{n=0}^{\infty} (A_{n+1}-A_n) f(A_n) \quad (D)$$

1.5.6 Ermakof's Test

Let $\varphi\colon [0,\infty) \to [0,\infty)$ be increasing with $\varphi(x) > x$, and $f\colon [0,\infty) \to [0,\infty)$ be nonincreasing. Then

$$\limsup_{x\to\infty} \frac{\varphi'(x) f(\varphi(x))}{f(x)} < 1 \implies \sum_{n=0}^{\infty} f(n) \quad (C)$$
$$\exists a > 0 : \inf_{x>a} \frac{\varphi'(x) f(\varphi(x))}{f(x)} \geqslant 1 \implies \sum_{n=0}^{\infty} f(n) \quad (D)$$

2 For Conditionally Convergent Series

2.0.1 Alternating Series

$$a_n \downarrow 0 \implies \sum_{n=0}^{\infty} (-1)^n a_n \quad (C)$$

2.1 For product series

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} a_n \quad |C| \\ &(b_n)_{n=0}^{\infty} \text{ bounded} \end{aligned}\right\} \implies \sum_{n=0}^{\infty} a_n b_n \quad |C|$$

2.1.1 Abel

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} a_n \quad (C) \\ &(b_n)_{n=0}^{\infty} \text{ monotone, bounded} \end{aligned}\right\} \implies \sum_{n=0}^{\infty} a_n b_n \quad (C)$$

2.1.2 Dirichlet

$$\left.\begin{aligned} &\sum_{n=0}^{k} a_n \text{ bounded} \\ &(b_n)_{n=0}^{\infty} \text{ monotone,} \\ &b_n \to 0 \end{aligned}\right\} \implies \sum_{n=0}^{\infty} a_n b_n \quad (C)$$

2.1.3 Du Bois-Reymond

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} a_n \quad (C) \\ &\sum_{n=0}^{\infty} (b_n - b_{n+1}) \quad |C| \end{aligned}\right\} \implies \sum_{n=0}^{\infty} a_n b_n \quad (C)$$

2.1.4 Dedekind

$$\left.\begin{aligned} &\sum_{n=0}^{k} a_n \text{ bounded} \\ &\sum_{n=0}^{\infty} (b_n - b_{n+1}) \quad |C| \\ &b_n \to 0 \end{aligned}\right\} \implies \sum_{n=0}^{\infty} a_n b_n \quad (C)$$