The Fundamental Theorems of Calculus

Richard Chapling

v1.0   23 May 2020

The plural is deliberate. A *Fundamental Theorem of Calculus*[1] is a result that simplifies a composition of derivative and integral operators. Since there are two orders in which we might compose them (namely *DI* and *ID*, schematically), there must be two fundamental theorems.

1 The basic versions

Given a function f, we recall that an *antiderivative* of f is a function F that satisfies $F' = f$ (everywhere). Since an antiderivative must be differentiable, it follows immediately that

Lemma 1. *An antiderivative is continuous.*

Theorem 2 (Fundamental Theorem of Calculus, DI). *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Define a function $F\colon [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f.$$

Then $F' = f$.

This says

1. Integrals of continuous functions are continuously differentiable, or
2. Any continuous function has an antiderivative, or
3. The operator $I(f) = \int_a^- f$ maps continuous functions to continuously differentiable functions, and on this space, the operator $D\colon f \mapsto f'$ is left-inverse to it; that is,

   $$C[a,b] \xrightarrow{I} C^1[a,b] \xrightarrow{D} C[a,b],$$

   with $DI = \mathrm{id}$.

The idea of the proof of this result is very simple, and only requires us to use the definitions, and some of the basic integral inequalities and identities.

Proof. First we show F is continuous. Let $x \in [a,b]$. For h sufficiently small, we have

$$F(x+h) - F(x) = \int_x^{x+h} f,$$

[1] The terminology is notably quite new, credited to Paul du Bois-Reymond.[2] This reflects changing attitudes to what is important in the calculus over time: the result meant something quite different before Cauchy's definition of the integral using sums.

[2] German, not French, by the way. His family were Huguenots, and emigrated to Germany several generations prior. Probably the most important mathematician of the late nineteenth century you've never heard of.

by the additivity of the integral. Since f is bounded, by K, say, we immediately have

$$|F(x+h) - F(x)| = \left| \int_x^{x+h} f \right| \leqslant |h| K,$$

by the simplest integral inequality $\left| \int_a^b f \right| \leqslant (b-a) \sup_{[a,b]} f$. Hence F is continuous on $[a, b]$ (the endpoints are slightly different, in that we only need to consider positive or negative h, but the method is the same).

Now suppose that $x \in (a, b)$. We need to show that

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f = f(x). \tag{1}$$

Let $\varepsilon > 0$ and $h \neq 0$. Since f is continuous at x, we can find $\delta > 0$ so that $|f(y) - f(x)| < \varepsilon$, for all y with $|y - x| < \delta$. But the integral is positive and linear, so preserves inequalities, and thus if $|h| < \delta$, we have

$$\left| \frac{1}{h} \int_x^{x+h} f - f(x) \right| = \left| \frac{1}{h} \int_x^{x+h} (f(y) - f(x)) \, dy \right| \leqslant \frac{1}{|h|} \int_x^{x+h} |f(y) - f(x)| \, dx < \frac{|h|}{|h|} \varepsilon = \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we immediately conclude that (1) holds. □

Sometimes the following alternative proof is offered:

Alternative proof of Theorem 2. If f is continuous, by the Mean Value Theorem for Integrals there is $\eta = \eta(h)$ strictly between 0 and h so that

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f = f(x + \eta(h)).$$

Since $\eta(h)$ is between 0 and h, it converges to 0 as $h \to 0$, and since f is continuous, $f(x + \eta(h)) \to f(x)$ as $h \to 0$. □

There are two reasons that our first proof is better: the Mean Value Theorem for Integrals requires doing part of the first proof anyway, and this also needs that f is continuous on a neighbourhood of x: as we note later, this is not necessary, and the first proof gives a more general result.

The other fundamental theorem is usually given as

Theorem 3 (Fundamental Theorem of Calculus, ID). *Let $G \colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and continuously differentiable on (a, b), with $G' = g$. Then*

$$\int_a^b g = G(b) - G(a). \tag{2}$$

This says

1. Integrals can be computed using antiderivatives, or
2. The integral of a (sufficiently nice) derivative is given by difference of the original function evaluated at the endpoints.

The proof of this result is rather more difficult, and (not surprisingly), we need the Mean Value Theorem.

Proof. Let $P = \{x_0 = a, x_1, \dots, x_n = b\}$ be a partition of $[a, b]$. We can write

$$G(b) - G(a) = \sum_{k=1}^{n} (G(x_k) - G(x_{k-1})).$$

Now, G is continuous on $[x_{k-1}, x_k]$ and differentiable on (x_{k-1}, x_k) for each $k \in \{1, \dots, n\}$, so we can apply the Mean Value Theorem to find $t_k \in (x_{k-1}, x_k)$ so that

$$G(x_k) - G(x_{k-1}) = g(t_k)(x_k - x_{k-1}).$$

Inserting these into the sum gives

$$G(b) - G(a) = \sum_{k=1}^{n} g(t_k)(x_k - x_{k-1}).$$

Since $\inf_{[x_{k-1},x_k]} g \leqslant g(t_k) \leqslant \sup_{[x_{k-1},x_k]} g$ for any $t_k \in [x_{k-1}, x_k]$, we immediately have

$$L(g; P) \leqslant G(b) - G(a) = \sum_{k=1}^{n} g(t_k)(x_k - x_{k-1}) \leqslant U(g; P),$$

for any partition P, and hence, taking the supremum over all P of the left inequality and the infimum over P of the right inequality,

$$\underline{\int_a^b} g \leqslant G(b) - G(a) \leqslant \overline{\int_a^b} g.$$

Since g is integrable, the left and the right sides are both equal to $\int_a^b g$, so the centre is too. □

However, because we assumed that g was continuous, a much simpler proof is also available to us:

Cheap proof. g is continuous. Hence $H(x) := \int_a^x g$ exists, and by Theorem 2, $H - G$ has derivative 0. Therefore by the Constant Value Theorem, $H - G$ is constant, and evaluating at a shows that $H(x) - G(x) = -G(a)$, so $H(b) = \int_a^b g = G(b) - G(a)$. □

This proof is often used by books in a hurry to "get to the good stuff", since it covers the most common cases, but often g being continuous is too much to ask.

2 Improvements

Scrutinising the proof of Theorem 2, we see that we actually only used that f was continuous at the point x; elsewhere, the continuity is simply used to ensure that f was integrable (i.e. so that F could be defined in the first place).

Theorem 4 (Fundamental Theorem of Calculus, DI). *Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Define a function $F : [a, b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f.$$

Then

1. *F is continuous, and*
2. *if f is continuous at $x \in (a, b)$, F is differentiable at x with $F'(x) = f(x)$.*

Similarly, since our version of the Mean Value Theorem only requires that g' exist, rather than be continuous, we can immediately refine Theorem 3 into

Theorem 5 (Fundamental Theorem of Calculus, ID). *Let $G\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), with $G' = g$ integrable. Then*

$$\int_a^b g = G(b) - G(a).$$

Since g' is no longer continuous, there is no "cheap" way of proving this as a corollary of Theorem 2.

Remark 6 (Further generalisations). A natural question is how far one can extend these results. The answer for the Riemann integral is *not very*: there is no nice condition that tells us whether FToC (ID) will work for a function, that is if a function is an antiderivative of an integrable function; one of the prime motivations for the development of a new integral at the end of the nineteenth century was this inadequacy in the Riemann integral.[3] One thing we *can* do is relax the requirement that $G' = g$ *everywhere* to $G' = g$ *everywhere apart from a finite set*. This is straightforward enough by partitioning the interval at the points of inequality *before* doing the usual proof, and then recombining everything at the end.

It is not clear what a sensible generalisation of FToC (DI) should be, since it is obvious that if f is not continuous at x, we should not expect any limits involving points nearby to converge to it. There are more general results available if we are prepared to ask for "less" differentiability: for example, if we allow f to only be continuous on the right (or left) at x, we find that F is differentiable on the right (or left). A much larger generalisation is the Lebesgue Differentiation Theorem, which says, among other things, that the FToC (DI) is true "almost everywhere"[4] for Lebesgue-integrable functions (which include Riemann-integrable functions).

3 A useful consequence

We can adapt the proof of Theorem 2 to show that continuous functions are integrable without using a contradiction or any uniform ideas:

Proposition 7. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is integrable.*

Proof. f is continuous, so it is bounded. Hence the upper and lower integrals exist. Let $\varepsilon > 0$, find δ so that $|f(y) - f(x)| < \varepsilon$ for every y with $|y - x| < \delta$. Then for $0 < h < \delta$, we have

$$f(x) - \varepsilon \leqslant \frac{1}{h}\underline{\int}_x^{x+h} f \leqslant \frac{1}{h}\overline{\int}_x^{x+h} f \leqslant f(x) + \varepsilon$$

Writing $L(x) = \underline{\int}_a^x f$ and $U(x) = \overline{\int}_a^x f$, we therefore have

$$f(x) - \varepsilon \leqslant \frac{L(x+h) - L(x)}{h} \leqslant \frac{U(x+h) - U(x)}{h} \leqslant f(x) + \varepsilon.$$

Taking $h \to 0$, we see as before that $U'(x) = L'(x) = f(x)$. But then $(U - L)' = 0$, and $U(a) = L(a) = 0$, so the Constant Value Theorem implies that $U(b) = L(b)$, so the upper and lower integrals are equal and f is integrable. □

[3] See Hawkins, T. *Lebesgue's Theory of Integration: Its Origins and Development*, which I cannot recommend too highly. It turns out that there *is* a simple condition ("absolute continuity") for a function to be a primitive of a Lebesgue-integrable function. There is a still more general integral called the gauge, or Henstock–Kurzweil integral, which can actually integrate *any* derivative.

[4] In a precise sense defined using Lebesgue measure: a property holds almost everywhere if the set where it does not hold can be covered by a countable set of intervals of total length as small as we like.

4 Nomenclature

Mathematicians are often very bad at naming things (for example, after the wrong people). This is often because the results are named and reused before their true significance is properly understood. We do not have this excuse here. Many books (and many mathematicians) use completely unhelpful convention of refering to the two theorems as FToC 1 and FToC 2 (if they even name both): a large list of undergraduate analysis textbooks and their conventions is given in Table 1.[5] This may not seem problematic, until I point out that *there is no logical ordering between the two*: they can be presented equally well in either order, depending on your point of view:

1. If you are interested in the integral as a function of its endpoints, Theorem 2 is a natural result to try to prove. Theorem 3 can then be derived as a trivial (although useless) corollary.
2. But if you are interested in evaluating integrals, it is natural to give Theorem 3 first, then show that we actually have antiderivatives for the functions we normally care about, namely the continuous ones. So Theorem 3 would be FToC 1.
3. If you think that they are talking about the derivative being a left and right inverse on the class of continuous functions, you don't care about the order.

I therefore decided to give the results the designations DI and ID, since these are both unambiguous and illustrative of their content. I have so far found exactly four books that make the distinction in this way (rather than just arbitrary labelling): Lewin's *An Interactive Introduction to Analysis*, Strichartz's *The Way of Analysis*, Schröder's *Mathematical Analysis: A Concise Introduction*, and Thompson, Bruckner and Bruckner's *Elementary Real Analysis*.

Table 1: Naming conventions for the Fundamental Theorem of Calculus (abbreviated FToC). Only post-war books written in English which consider definitions equivalent to the Darboux integral (i.e. that taught in ANALYSIS I) are included.[6] "[]" denotes editorial comment not explicitly quoted from the text.

Book	Theorem 2	Theorem 3	Theorem 4	Theorem 5
Abbott, S. *Understanding Analysis*			Theorem 7.5.1 (FToC) (ii)	Theorem 7.5.1 (FToC) (i)
Alcock, L. *How to Think about Analysis*	§ 9.8 FToC			
Al-Gwaiz, M. A. and Elsanousi, S. A. *Elements of Real Analysis*	Corollary 8.11 [unnamed]		Theorem 8.11 [unnamed]	Theorem 8.12 (FToC)
Amann, H. and Escher, J. *Analysis II*	4.12 Theorem (of the differentiability in the entire interval) [First FToC]	4.13 Theorem (Second FToC)		
Apelian, C. and Surace, S. *Real and Complex Analysis*			Theorem 2.15 [unnamed]	Theorem 2.17 (FToC)
Apostol, T. *Calculus, Vol. 1*		Theorem 5.3 Second FToC	Theorem 5.1 First FToC	
Banner, A. *The Calculus Lifesaver*	First FToC	Second FToC		
Ball, D. G. *An Introduction to Real Analysis*	Theorem 8.4.4 (the fundamental theorem of integral calculus)	Corollary 2 [unnamed]		
Bartle, R. G. and Sherbert, D. R. *Introduction to Real Analysis*	7.3.6 Theorem [unnamed]		7.3.5 FToC (Second Form)	7.3.1 FToC (First Form)[7]

[5] Worse, there is no agreement on which set of hypotheses are included, so FToC 1 and 2 can refer to any of the *four* theorems we have discussed.

[7] The result proved is more general: they allow $G' \neq g$ on a finite set.

Book	Theorem 2	Theorem 3	Theorem 4	Theorem 5
Bashirov, A. *Real Analysis Fundamentals*			Theorem 9.26 (FToC) (b)	Theorem 9.26 (FToC) (a)
Bauldry, W. C. *Introduction to Real Analysis: An Educational Approach*	Theorem 1.28 (FToC) 1.	Theorem 1.28 (FToC) 2.		Theorem 2.42 (FToC)
Beals, R. *Analysis: An Introduction*	Theorem 8.20 (Differentiation of the integral)	Corollary 8.21 (FToC)		
Berberian, S. K. *A First Course in Real Analysis*	9.4.6 Theorem (FToC) (2)[8]	9.4.7 Corollary [unnamed]		
Bilodeau, G. G. and Thie, P. R. and Keough, G. E. *An Introduction to Analysis*	Theorem 5.4.1 (FToC) (a)	Theorem 5.4.1 (FToC) (b)		
Bloch, E. D. *The Real Numbers and Real Analysis*			Theorem 5.6.2 FToC Version I	Theorem 5.6.4 FToC Version II
Brand, L. *Advanced Calculus: An Introduction to Classical Analysis*			Theorem 119.2 [unnamed]	§ 120 Fundamental Theorem of the Integral Calculus
Brannan, D. *A First Course in Mathematical Analysis*	Theorem 7.4.4 [unnamed][9]			Theorem 7.3.1 (FToC)
Bridger, M. *Real Analysis: A Constructive Approach*	Theorem 6.3.1 (FToC, Part I)[10]	Theorem 6.3. (FToC, Part II)[10]		
Bryant, V. *Yet Another Introduction to Analysis*	Cor. (FToC)			
Burkill, J. C. *A First Course in Mathematical Analysis*		Theorem 7.63 [unnamed]	Theorem 7.62 [unnamed]	
Burn, R. A. *Numbers and Functions: Steps into Analysis*	qn 54 [FToC]		qn 56 [unnamed]	
Carlson, R. *A Concrete Introduction to Real Analysis*	Theorem 6.3.4 (FToC: Part 1)	Theorem 6.3.5 (FToC: Part 2)		
Chatterjee, D. *Real Analysis*	Theorem 6.5.1 [unnamed]			Theorem 6.5.2 (Fundamental Theorem of Integral Calculus)
Conway, J. B. *A First Course in Analysis*		3.2.2 Corollary [unnamed]	3.2.1 Theorem FToC	
Cummings, J. *Real Analysis: A Long-Form Mathematics Textbook*	Theorem 8.32 (FToC) (ii)			Theorem 8.32 (FToC) (i)
Delinger, C. G. *Elements of Real Analysis*			Theorem 7.6.8 (FToC, Second Form)	Theorem 7.6.2 (FToC, First Form)
Dutta, H. and Natarajan, P. N. and Cho, Y. J. *Concise Introduction to Basic Real Analysis*	Theorem 8.3.2 (FToC I)			Theorem 8.3.3 (FToC II)
Duren, P. L. *Invitation to Classical Analysis*			FToC (a)	FToC (b)
Field, M. *Essential Real Analysis*	Noted in Remark 2.8.7[11]	Lemma 2.8.5 [unnamed][11]		
Fischer, E. *Intermediate Real Analysis*	Corollary 2 (of Theorem XIII.6.2)[12]	Corollary (of Theorem 6.1)	Corollary 1 (of Theorem XIII.6.2)[12]	Theorem XIII.6.1 (FToC)

[8] In the form "there exists a continuous function $F\colon [a,b] \to \mathbb{R}$, differentiable on (a,b), so that $F'(x) = f(x)$ for all $x \in (a,b)$"

[9] "closely related to the [FToC]"

[10] The results are weaker, in accordance with the book's constructive approach: they use uniform continuity and uniform differentiability.

[11] Field's presentation is very unusual, defining the integral as something that satisfies two properties of betweenness and additivity. Hence the theorems referenced here do not correspond exactly to the usual scheme.

[12] Theorem XIII.6.2 Is the general case of left- or right-continuity implying left- or right- derivatives existing respectively. None of

Book	Theorem 2	Theorem 3	Theorem 4	Theorem 5
Fulks, W. *Advanced Calculus: An Introduction to Analysis*		4.5a Theorem [named as FToC in text]	[Consequence of] 4.2h Lemma [unnamed]	4.5b Theorem [unnamed]
Garling, D. J. H. *A Course in Mathematical Analysis Vol. I*			Prop. 8.5.2 (FToC) (i)	Prop. 8.5.2 (FToC) (ii)
Gaughan, E. D. *Introduction to Analysis*		Unnamed Theorem	5.14 Theorem [unnamed]	5.8 Fundamental Theorem of Integral Calculus
Ghorpade, S. R. and Limaye, B. V. *A Course in Calculus and Real Analysis*			Prop. 6.24 (FToC) (i)	Prop. 6.24 (FToC) (ii)
Haggarty, R. *Fundamentals of Mathematical Analysis*	7.11.1 FToC	explained, not named		
Hart, M. and Towers, D. *Guide to Analysis*	Theorem 6.4.4 FToC		Theorem 6.4.4(a) [unnamed]	
Hoskins, R. F. *Standard and Nonstandard Analysis*	Theorem 7.4 ["usually described as the FToC"]	[Discussed in § 7.2.2, unnamed]		Exercise 7.2.3 (version of FToC)
Howie, J. M. *Real Analysis*	Theorem 5.18 (FToC)	Theorem 5.20 [unnamed]		
Howland, J. *Basic Real Analysis*	Theorem 11 (FToC I) (?)	Corollary 5.8.3 [unnamed]		Theorem 5.8.2 (FToC II)
Jacob, N. and Evans, K. P. *A Course in Analysis, Vol, I: Introductory calculus Analysls of Functions of One Real Variable*	Theorem 26.1 [unnamed]	Theorem 26.2 (FToC)		
Johnsonbaugh, R. and Pfaffenberger, W. E. *Foundations of Mathematical Analysis*		Corollary 56.2 (ii)	Theorem 56.1 (Fundamental Theorem of Integral Calculus) (ii)	Theorem 56.1 (Fundamental Theorem of Integral Calculus) (i)
Junghenn, H. D. *A Course in Real Analysis*	5.3.1 FToC (a)	5.3.1 FToC (b)		5.3.1 FToC (c)
Keisler, H. K. *Elementary Calculus: An Infinitesimal Approach*	FToC (i)	FToC (ii)		
Kirkwood, J. *Introduction to Analysis*	Theorem 6-18 [unnamed]			Theorem 6-19 (FToC)
Knapp, A. W. *Basic Real Analysis*	Theorem 1.32 (FToC) (a)	Theorem 1.32 (FToC) (b)		
Kopp, E. *Analysis*	§ 11.1 Theorem 1 [FToC]	§ 11.1 Theorem 1 [FToC]		
Körner, T. W. *A Companion to Analysis*	Theorem 8.37 FToC	Theorem 8.42[13]		
Krantz, S. G. *Foundations of Analysis*	Theorem 7.21 (FToC)	Corollary 7.22 [unnamed]		
Krantz, S. G. *Real Analysis and Foundations*	Theorem 7.21 (FToC)	Corollary 7.22 [unnamed]		
Kumar, A. and Kumaresan, S. *A Basic Course in Real Analysis*		covered in Remark 6.3.6	Theorem 6.3.4 (Second FToC)	Theorem 6.3.1 (First FToC)
Lárusson, F. *Lectures on Real Analysis*			7.12 Theorem (FToC) (2)	7.12 (FToC) (1)
Lay, S. R. *Analysis with an Introduction to Proof*			Ch. 7 3.1 Theorem (FToC I)	Ch. 7 3.5 Theorem (FToC II)
Lewin, J. *An Interactive Introduction to Analysis*			11.12.2 Differentiating an Integral	11.12.3 Integrating a Derivative

these results are named

[13]Noted as another form in the text, not named explicitly.

Book	Theorem 2	Theorem 3	Theorem 4	Theorem 5
Little, C. H. C. and Teo, K. L. and van Brunt, B. *Real Analysis via Sequences and Series*	Theorem 5.7.1 (FToC)	Theorem 5.7.1 (FToC)		Theorem 5.7.2 [unnamed]
Loed, P. A. and Wolff, M. P. H. *Nonstandard Analysis for the Working Mathematician*	Theorem 1.11.3 [unnamed]	Theorem 1.11.7 (FToC)		
Malik, S. C. and Arora, S. *Mathematical Analysis*			Ch. 9 Theorem 16 [First FToC in the text]	Ch. 9 Theorem 17 [Second FToC in the text]
Morgan, F. *Real Analysis*	16.1 FToC I.	16.1 FToC II.		
Nicolaescu, L. I. *Introduction to Real Analysis*	Theorem 9.42 [unnamed]	Corollary 9.44 (FToC: Part II)		Theorem 9.43 (FToC: Part I)
Ok, E. A. *Real Analysis with Economic Applications*	FToC [14]	FToC [14]		
Pedrick, G. *A First Course in Analysis*			Theorem on p. 187 [later referred to as "second FT"] [15]	Theorem on p. 186 [later referred to as "first FT"]
Philips, E. G. *A Course of Analysis*	§ 7.4 [16] Theorem 2	§ 7.4 Theorem 3		
Ponnusamy, S. *Foundations of Mathematical Analysis*		Corollary 6.40 [unnamed]	Theorem 6.36 (The second FToC)	Theorem 6.35 (The first FToC)
Pons, M. A. *Real Analysis for the Undergraduate: With an Invitation to Functional Analysis*			Theorem 7.3.1 (FToC) (a)	Theorem 7.3.1 (FToC) (b)
Protter, M. H. *Basic Elements of Real Analysis*	Theorem 5.6 (FToC—first form)	Theorem 5.7 (FToC—second form)		
Protter, M. H. and Ewing, J. H. *A First Course in Real Analysis*	Theorem 5.7 (FToC—first form)	Theorem 5.8 (FToC—second form)		
Pugh, C. C. *Real Mathematical Analysis*			3.34 FToC	3.37 Antiderivative Theorem
Rankin, R. A. *An Introduction to Mathematical Analysis*	Theorem 30.2.2 [unnamed]			Theorem 30.2.3 (FToC)
Reade, J. B. *An Introduction to Mathematical Analysis*	6.29 Theorem [17]	6.22 Theorem [17, 18]		
Reed, M. C. *Fundamental Ideas of Analysis*	Theorem 4.2.5 (FToC, Part II)	Theorem 4.2.4 (FToC, Part I)		
Rosenlicht *Introduction to Analysis*	Theorem it § VI.4 (FToC)	Corollary VI.4.2 [unnamed]		
Ross, K. A. *Elementary Analysis: The Theory of Calculus*			34.3 FToC II	34.1 FToC I
Rudin, W. *Principles of Mathematical Analysis*			6.20 Theorem [unnamed]	6.21 FToC
Schröder, B. S. W. *Mathematical Analysis: A Concise Introduction*			Theorem 8.17 FToC, Derivative Form	Theorem 5.23 FToC, Antiderivative Form
Schramm, M. J. *Introduction to Real Analysis*	Theorem 17.14 (FToC)		Theorem 17.16 (Second FT)	

[14] Given as: if $f, F\colon [a,b] \rightrightarrows \mathbb{R}$ and f is continuous, then $F(x) = F(a) + \int_a^x f$ iff F is C^1 and $F' = f$

[15] "The term Fundamental Theorem of Calculus is applied to theorems asserting that differentiation and integration are operations on functions which 'reverse' one another." One therefore suspects Pedrick intends the labels, such as they are, to be merely positional rather than prescriptive.

[16] Entitled *The Fundamental Theorem of the Integral Calculus*

[17] The chapter describes both together as the FToC.

[18] The proof given actually covers Theorem 5, too.

Book	Theorem 2	Theorem 3	Theorem 4	Theorem 5
Sohrab, H. H. *Basic Real Analysis*			Theorem 7.5.8 (Second FT)	Theorem 7.5.3 (First FT)
Speight, J. M. *A Sequential Introduction to Real Analysis*	Theorem 11.25 (FToC version 1)	Theorem 11.26 (FToC version 2)		
Spivak, M. *Calculus*	Theorem 14.1 First FToC	Corollary	Mentioned informally	Theorem 14.2 Second FToC
Stahl, S. *Real Analysis: A Historical Approach*		Theorem 15.3.12 [unnamed][19]	Theorem 15.3.11 (FToC I)	
Stillwell, J. M. *The Real Numbers: An Introduction to Set Theory and Analysis*	4.8.1 FToC			
Stirling, D. *Mathematical Analysis and Proof*	Theorem 13.12 (FToC) [First part]			Theorem 13.12 (FToC) [Second part]
Stoll, M. *Introduction to Real Analysis*	Mentioned in Remarks after 6.3.4	Mentioned in Remarks after 6.3.4	6.3.4 Theorem (FToC)	6.3.2 Theorem (FToC)
Strichartz, R. S. *The Way of Analysis*	Theorem 6.1.2 (Differentiation of the Integral)	Theorem 6.1.3 (Integration of the Derivative)		
Tao, T. *Analysis I*			Theorem 11.9.1 (First FToC)	Theorem 11.9.4 (Second FToC)
Taylor, J. L. *Foundations of Analysis*			Second Fundamental Theorem (Theorem 5.3.3)	First Fundamental Theorem (Theorem 5.3.1)
Thompson, B. S. and Bruckner, J. B. and Bruckner, A. M. *Elementary Real Analysis*	8.8 (Differentiation of the Indefinite Integral)	8.9 (Integral of a Derivative)	8.26 (Differentiation of the Indefinite Integral)	8.27 (Integral of a Derivative)
Yau, D. *A First Course in Analysis*		Theorem 6.9 [FToC][20]	Theorem 6.12 [Derivative of the Integral, called the first FToC in som books]	
Zorich, I. *Mathematical Analysis I*		Theorem 6.3.2 (Newton–Leibniz formula, or FToC)[21]	Lemma 6.3.1 [unnamed]	
Zorn, P. *Understanding Real Analysis*	Theorem 5.20 (FToC, version 2)	Theorem 5.19 (FToC, version 1)		Theorem 5.9 (FToC, version 0)

Lastly, we note that in E. Goursat's influential *Cours d'analyse mathématiques*, from the première édition on, (2) is called "la formule fondamental" (p. 173 of 1ère édition), while from the deuxième édition onwards, Theorem 2 is called "le théorème fondamental du Calcul intégral" (p. 185 of 2ème édition).

[19] Yes, this book really does have no FToC II!
[20] The proof given actually works as a proof of Theorem 5.
[21] The result is slightly more general: a finite number of discontinuities are allowed.