Introduction to Analysis I

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Welcome to Analysis I. Some people like to say that this is the dullest course in IA. They are completely wrong (and not just because of the existence of Vectors and Matrices). To disabuse you of this notion, let's have a look at what this course is for, and why we should want to do it.

What do we want to do? Unlike many other courses, this course has a clear goal: unlike Groups being to learn about groups, or Differential Equations giving you a bunch of techniques that we promise you'll need later, what we need to do is understand functions to and from the real numbers, to justify the various parts of the calculus (indeed, all of the calculus, eventually).

Why do we want to do this? The first sentence of the Analysis I schedule used to be "The need for rigour in calculus". Why do we need to rigorise calculus? On one level, this is a silly question: we're doing Mathematics, and Mathematics is about being sure about things. The mathematical way to be sure about things is to prove them, and this is what we mean by rigour. Hence if calculus is to be a part of Mathematics, rather than witchcraft or string theory, we need to treat it rigorously.

A natural next question is "is what we have done so far not rigorous, then?". After all, I'm sure you've written the word "Proof" in your working when producing results using calculus before, and probably some $\implies$ symbols as well. So you have been writing things that have the *form* of a proof. But are they actually a proof, i.e., does the conclusion always follow? Of course not, results are almost never true of any mathematical object: you need to start with some objects, and these objects need to have properties, which are called the *premises* of a theorem. The other way to pick a hole in a proof is to determine that one of the $\implies$ is not valid, so the other problem is *what counts as a valid deduction in a framework that is sufficient to give us calculus.*

Therefore we have now obtained what Analysis should be: we want useful results, like "when the function has a local maximum, the derivative vanishes", or "the derivative of the integral is the original function". Mathematical Analysis is the process of finding the correct conditions for conclusions of this type (i.e. "calculus" results) to be true, and then proving them based on these premises.

However there is another extremely common and pernicious argument for it not being necessary to rigorise the calculus.

But it works? We now tackle that most pernicious of excuses for not doing Analysis: calculus works. But of course, you say, if it didn't work, we wouldn't want to prove it! True, but this is not actually what I mean here. Calculus "works" in the sense that if you followed the instructions correctly, every question you answered using calculus gave a correct-looking answer. Therefore, the reasoning goes, there is no need to prove anything differently from the methods we have used before. This is the fallacy we might call the tiger-repellant socks effect.[1] Applied mathematicians, physicists, economists, and so on,[2] all, in general, want you to be *applying* mathematics to *other things*, rather than actually studying Mathematics-in-itself (AKA Pure Mathematics). They thus try to convince you that the Mathematics is easy and easily justified, and the way they do this is to bring you up on a bland diet of functions that behave so politely and meekly that the operations of calculus are made to seem banal, easy and universally applicable. (This is in fact very much in the tradition of Lagrange: I set up my definition of function to be sufficiently restrictive that the all of the justifications of my results are simple.) This is like saying that a sheep is a black woolly mammal. Then all our results apply to black sheep, and then along comes a white one, and we don't even know where to start. (And worse, at least in this part of the world, most sheep are white.) Much better to start with the correct definition of sheep, and base our proofs on its sheepiness, rather than its blackness. (Or, even better, on its being a mammal where possible.)

[1] My socks are tiger-repellant. Proof: You don't see any tigers in the room, do you?
[2] Compscis form a notable exception: they eventually realised that doing actual mathematics was properly useful to them.

So what shall we do? When you did your first course in number theory, you introduce the basic object you're working with, namely the integers. You then augmented this with additional objects like the integers modulo n, definitions like prime number, techniques like Euclid's algorithm, and so on.[3] This is the standard mathematical way of building up a subject.

Therefore, in this course we shall do the same. Unlike Numbers and Sets (and Groups), though, we know where we want to go: while it is difficult to pose questions in Number Theory without knowing what Number Theory is, it is quite easy to pose questions that require calculus without knowing about the real numbers.

A significant difficulty is that the real numbers are a much more complicated object than $\mathbb{Z}$, even on a basic level as a set: you know that they are uncountable, for one thing. It is hence not desirable to actually *construct* the real numbers as an object, at least at the beginning of a first course. It is more useful to give an *operational* definition, which is the "right" sort of definition to use in mathematics.[4] Just as we said that the natural numbers are any object that satisfies Peano's axioms, but we showed neither existence nor uniqueness for the natural numbers in Numbers and Sets, the real numbers are defined to be a totally ordered field containing the integers, with an extra property, called *Completeness*. As is often the case in Mathematics there are many formulations of this property that lead to the same object, but again, a first course is not the time to discuss this. A standard form of the completeness property to take is the Least Upper Bound property:

> Any nonempty set of real numbers with an upper bound has a least upper bound: the set $\{r \in \mathbb{R} : \forall s \in S, s \leqslant r\}$ has a least element.

(S itself need not have either a least or greatest element, even if it is bounded: $\{(-1)^n(1-2^{-n}) : n \in \mathbb{N}\}$ has neither a least nor a greatest element.)

The rest of this course will explore the consequences of this property, if you want to be an oblivious mathematician. For the rest of us, the rest of this course will use this property to show that the real numbers are the "correct" place to do calculus, and derive from it many of the results you know and love, together with many new friends whose company you have not yet made. The journey continues in more dimensions next year in Analysis II. Later this year, in Metric and Topological Spaces, you will see what happens when we take away some of the properties of the real numbers that make the theory in this course relatively well-behaved.[6]

Some general waffly philosophical remarks If we forget that the real numbers are a field for a moment, notice that, as an ordered space, everywhere looks pretty much the same, and there are no interesting individual points. This is essentially the complete opposite of the natural numbers' structure, where every number looks completely different, and each one has some or other strange unique property (recall the infamous seemingly-random natural number 1729).

Putting the field structure back on $\mathbb{R}$, it looks like 0 and 1 should be important numbers, since they appear in the axioms. In fact, 1, while quite important, is nowhere near as significant in the theory as 0, because dividing by zero is what we can't do, but really want to do all over the place in calculus. Indeed, you could say that the entire course will be about being sufficiently close to 0.[7]

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[3] Similarly, in Groups, you defined a group, then proved a load of results about groups, defined some useful ancillary notions like group action, homomorphism, and normal subgroup, proved results about these, and so on.

[4] An example you should be familiar with from Vectors and Matrices: it is easier to prove things about linear maps than about matrices.[5]

[5] On the other hand, it is often easier to do calculations with matrices. Likewise, while it is easier to prove things about real numbers using the definition we give below, it is easier to do calculations with decimal expansions. The moral of the story is that it is best to have several descriptions of an object.

[6] Not that real analysis is not complicated and difficult; it's just rather easier than what remains when you remove a few of the properties of $\mathbb{R}$, like ordering, or being homogeneous, or even having a function that measures the distance between points.

[7] After these, the next most useful real number is probably e; in this course, π plays very much second (if not third or fourth) fiddle. We shall meet e late in the course.

Books and resources

It is quite possible that this is the pure mathematics course than has the most textbooks available. The number is completely ridiculous, and this can be rather overwhelming for the inexperienced. Similarly, there are numerous resources available online for this course, or variations, in different institutions. I shall look at a few standouts in each category, and then give recommendations based on what you are looking for.

Books

Let's start with the recommended books in the schedules. There are several possible different orderings of the material, which result in different emphasis and different methods of introducing the elementary transcendental functions;[8] in each case this is noted. The ordering in the Cambridge course is normally sequences and series, continuity and differentiation, integration.

- T.M. Apostol *Calculus*, vol 1. Wiley 1967–69

  Almost absurdly comprehensive, returning to some topics many times with increasing rigour. It is thus quite difficult to give a hard and fast ordering, but most notably, it *begins* with integration, later covering continuity, differentiation, the elementary transcendental functions, Taylor series, and eventually sequences and series. The last third is mostly linear algebra and some other topics, including space curves, that are not relevant. The trigonometrical functions are defined using the area of a sector to define the angle and the appropriate ratios, the logarithm as an integral, the exponential as its inverse. Exercises are on the easy side. Its main detraction is the price, which is absurdly high for a book of this type.[9]

- J.C. Burkill *A First Course in Mathematical Analysis.* Cambridge University Press 1978

  A very short book that covers slightly more than the contents of the Schedules. Quite terse, and omits optional but important topics like limit inferior and limit superior, but unlike Rudin (see below), still quite readable, although some of the proofs are probably too abbreviated. Ordering is unusual, and covers sequences, continuity and derivatives, series, special functions (defined by series), integration (and a last chapter covering some of the basics of differentiation in the multivariate theory). Out of print.

- D.J.H.Garling *A Course in Mathematical Analysis (Vol 1).* Cambridge University Press 2013

  The style of this book is quite similar to many of the courses in Tripos, so I would recommend it based on that alone. The first Part contains an account of formal set theory and is optional; the relevant part for this course is the second. Ordering is sequences, series, continuity, differentiation, integration, as in the Schedules. The elementary transcendental functions are defined by power series. Only the first 240-odd pages is relevant to the course, but the last two chapters introduce other interesting topics. The second volume is appropriate for the Part IB courses Metric and Topological Spaces and Analysis II, while the last covers the contents of Complex Analysis (and a bit more) and Measure Theory.[10] Should be available as part of Cambridge Core at `https://www.cambridge.org/core/books/course-in-mathematical-analysis/C0D89CA72FF3ED2B7F3280A922CF9D5B`, to which Cambridge has a subscription.

- J.B. Reade *Introduction to Mathematical Analysis.* Oxford University Press

  The shortest book on this list. Some results are less general than in most books, presumably to shorten the proofs. Order is sequences, series, continuity, differentiation, integration, as in the Schedules. The elementary transcendental functions are defined via power series in the chapter on continuity. Expediency rather trumps rigour in certain places, while in others the proofs are both ingenious and simple. Out of print, but available (apparently with permission) at `http://www.mathstudio.co.uk/sequences.htm`.

- M. Spivak *Calculus.* Cambridge University Press 2006

  Now in its fourth edition. (Mostly the same as the third. Earlier editions have some errors). I learnt from this book. Explanations are good, and has some excellent challenging exercises, although it may be said that

[8] The exponential, logarithm, trigonometrical and cyclometrical functions, all of which *require* some limiting process be used to define them. Some books call these *special functions*, but this term usually also covers higher transcendental functions like Bessel functions that you will study next year.

[9] A word to the wise: most electronic copies have completely trashed typesetting.

[10] I suspect that its approach is not particularly compatible with the Part II course Probability and Measure.

too much of the interesting stuff has been relegated to the exercises.[11] Order is continuity, differentiation, integration, elementary transcendental functions, sequences and series. Trigonometrical functions are done using the area of the segment of a circle as basis, and the logarithm is defined using an integral. Ends with a thorough discussion of various ways of defining the real numbers, including existence and uniqueness.[12]

- David M. Bressoud *A Radical Approach to Real Analysis.* Mathematical Association of America Textbooks

  This is an excellent second book, that takes a much more historically-motivated[13] approach to topics covered in every book on this list. Also recommended for later is the sequel, *A Radical Approach to Lebesgue's Theory of Integration.*

Next, we look at a few other notable books.

- W. Rudin, *Principles of Mathematical Analysis*[14]

  This book is infamously challenging. Nominally, it is a first course in Analysis. It is recommended for Metric and Topological Spaces for a reason, namely that it begins with a general consideration of convergence and continuity on a metric space, from which the results on $\mathbb{R}$ emerge as special cases! Once you get past this, the rest of the book is a vivid example of mathematical exposition of a certain very arid and dense type that has an enduring popularity, despite (or perhaps because of) it requiring a lot of effort on the part of the reader.[15] The last part, on multivariate analysis, is also extremely brief and rather cursory, even by the standards of the rest of the book.

- T.W. Körner, *Companion to Analysis*

  If something in this course feels like it has been glossed over, or you want to understand why something is defined in a particular way, this book probably has the answer. The Appendices are especially valuable for their extensions of concepts and sage advice.

- S. Abbott, *Understanding Analysis*, Springer, 2016

  A different approach to the standard material: rather than the "justify calculus" approach, this book looks at the places where calculus results don't work, and we are forced to deploy Analysis to understand what is going on. Samples of various sections are available at `http://community.middlebury.edu/~abbott/UA/UA.html`.

Lastly, books that are historical or rather less mainstream.

- G.H. Hardy, *A Course of Pure Mathematics*

  The original analysis textbook in English, and one of the most important books in the history of Tripos. As is typical of Hardy, the prose is exquisite, the ideas conveyed clearly. Does show its age, and terminology has moved on somewhat since it was written. The treatment of the integral is also not up to modern standards.

- C. Jordan, *Cours d'analyse de L'École polytechnique* (3 vols), Paris, (3^e ed. 1909–13)

  This is in French. Go for the second or third edition: the first edition is not of the same class. I recommend this because it is "the real thing": a difficult, but inspirational, classic of the genre. Hardy writes

  > [...] it is fair to attribute to the inspiration of [this book] the beginnings of the movement which, carried on by Hadamard, Borel and Lebesgue, has revolutionised the foundations of modern analysis.[16]

  Hardy also notes

  > The book, for all its masterfulness, is by no means a very easy one to read. Jordan, I imagine, was no believer in easy roads to the understanding of mathematical truth; he could not shirk a difficulty himself, and he had no intention of allowing his readers to do so. [...]

[11] Including such useful results as the ratio test!
[12] The foreword to Pete L. Clark's course notes for "Honors Calculus", available here: `http://alpha.math.uga.edu/~pete/expositions2012.html` has a more extensive review.
[13] But not historically-minded; there is a difference.
[14] AKA "Baby Rudin", to distinguish it from Rudin's other two, more advanced, books.
[15] Much to my disappointment. Editors like it because it fells fewer trees.
[16] Ironically, none of this is included in Whittaker and Watson's book discussed above, despite its title.

The reader is encouraged to refer to Hardy's obituary of Jordan[17] to read the full glowing review. A digitised copy of the third edition of all three volumes is available at `https://archive.org/details/CoursDanalyseDeLcolePolytech`

- E.T. Whittaker and G.N. Watson, *Modern Analysis*, Cambridge University Press, 1927.[18]

  Do not read the first chapter on the real numbers: it is one of the poorest explanations I have ever seen. The first quarter of the book is a crash course in the techniques taught in Analysis I, Complex Analysis and Further Complex Methods, along with some topics (such as integral equations) that are no longer fashionable, while the rest is spent applying these to the more common special functions (Γ, the Riemann zeta-function, elliptic functions, &c.). Original sources cited throughout.

Recommendations:

If you want a challenge Rudin's *Principles of Mathematical Analysis*

If you want a classic Jordan's *Cours d'Analyse*

If you want a solid all-round book Spivak's *Calculus*

Other resources

- I'm sure you're all aware of Dexter Chua's notes. Bear in mind that they are based on Professor Gowers's interpretation of the material, which differs a fair amount from the standard presentation in places.
- Speaking of Professor Gowers, there are a number of posts available on his blog about this course, `https://gowers.wordpress.com/category/cambridge-teaching/ia-analysis/`.
- Another previous lecturer of note is Vicky Neale, who has sadly now moved to the Other Place.[19] After every lecture, she wrote a blog post containing a summary and topics to think about, which are found (in reverse order) on her (now inactive) `https://theoremoftheweek.wordpress.com/category/cambridge-maths-tripos/ia-analysis-i/`.

Further reading

We separate the books in this section into further reading on various topics suggested by the course, or alternative approaches that may interest you.

Series

- T.J.I'A. Bromwich, *An Introduction to the Theory of Infinite Series* (Second edition), Macmillan, 1926.

  A rather eclectic book notable for the first edition being essentially the first book on rigorous analysis published in England. The author clearly wanted to write a book to accompany his course on infinite series, but realised that it was necessary to elaborate on the other parts of Real Analysis to cover the theory properly, so a significant chunk of the book is spent on defining other notions of use, such as the elementary transcendental functions. The result is a rather inside-out Analysis I–level book, with a lot more in in than you need. Would be the definitive book on infinite series (at least in English) if not for the next item.

- K. Knopp, *Theory and Application of Infinite Series*, Second Edition, Blackie and Son, 1951 (Reprinted by Dover, 1990).

  Not to be confused with Knopp's other book about infinite series, which is rather shorter. This is probably still the definitive book about the general theory of infinite series as discussed in this course. Contains numerous references to the original literature. Translated from the original German *Theorie und Anwendung der unendlichen Reihen*.

[17] Proceedings of the Royal Society (A), Volume 104, Issue 728 (1923), xxiii–vi, available online at `https://royalsocietypublishing.org/doi/10.1098/rspa.1923.0126`
[18] Note both the title and the date
[19] And is very much alive, if you have an overactive interpretation of seemingly-morbid euphamism.

The Mean Value Theorem Yes, there are two books just on the Mean Value Theorem:

- P.K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific, 1998.

  This book is mostly about solving a large number of functional equations that arise from Mean Value Theorems of various types, but covers a wide variety of MVTs on the way: Lagrange's, Cauchy's, Pompeiu's and Flett's are treated at length, and even the obscure McLeod MVT is mentioned.

- C. Smoryński, *MVT: A Most Valuable Theorem*, Springer, 2017.

  A historically-presented walk through the development of the MVT, first putting all the ideas in place, then the developments from Lagrange, to Cauchy, Bonnet, Peano and into the twentieth century. Quotes original sources extensively in translation. Ends with a rant about calculus reform.

Alternative approaches

- H. Jerome Keisler, *Elementary Calculus: An Infinitesimal Approach*, Third Edition, Dover, 2012 and online.

  An approach to Real Analysis using Robinson's rigorous infinitesimals. This relies on a logical construction called a *nonprincipal ultrafilter*, but the book is deliberately written to be approached as a first course in Analysis. The author has made the most recently-revised edition available for free on his website, `https://www.math.wisc.edu/~keisler/calc.html`.

- Robert G. Bartle and Donald R. Sherbert, *Introduction to Real Analysis*, Fourth Edition, Wiley, 2011.

  This mostly looks like a standard textbook (apart from the rather odd placement of infinite series after integration, several chapters later than sequences). However, it develops the notion of a *gauge*: a positive function $\delta(x)$ that may be used both to simpify some proofs of the standard results, and to prove more general ones. This really comes into its own in Chapter 10, where a new integral, more general than the Riemann integral, is introduced. This is the Henstock–Kurzweil integral (also called the gauge integral); it has several wonderful properties: it includes all improper integrals already, and integrates every derivative. Its definition is made by replacing the δ in Riemann's definition of integral by a gauge function, a small change with a very large impact. (Indeed, this integral also includes the Lebesgue integral on $\mathbb{R}$!)

Integration [20]

- Eric Schechter, *An Introduction to the Gauge Integral*, `https://math.vanderbilt.edu/schectex/ccc/gauge/`, 2009

  A good list of resources on the gauge integral, including many books.

- J. Mikusiński, *The Bochner Integral*, Springer, 1968.

  Despite its name, this book is really about approaching the Lebesgue integral using a result published by the author, that essentially says a function is Lebesgue-integrable if it is almost everywhere the pointwise limit of an absolutely-convegent sequence of functions. It is terse, axiomatic, and occasionally very obscure, but requires no measure theory.

[20] The author admits that this section is perhaps even more reflective of his tastes than the others in this document.