The Root Test

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We give here a useful alternative to the ratio test.[1]

Theorem 1 (Basic root test). *Let $\sum_n a_n$ be a complex series. Suppose that $R := \lim_{n\to\infty} |a_n|^{1/n}$ exists. Then*

Convergence *If $R < 1$, $\sum_n a_n$ converges absolutely.*

Divergence *If $R > 1$, $\sum_n a_n$ diverges.*

No conclusion can be drawn if $R = 1$.

This is applicable in more situations than the ratio test (we don't have to divide by anything, so we don't have the issue with zeros that the ratio test has), and gives a result whenever the ratio test does. Its main disadvantage is that in many common examples the limit involved is harder to evaluate. The rule of thumb is that if the terms contain factorials, the Ratio Test (or a test using the ratio, anyway) is the way to go, while otherwise, the Root Test is usually easier.

The proof is straightforward: the idea is to use the geometric series in a "better" way than the ratio test does.

Proof. **Convergence** Suppose that $R < 1$. Then by the definition of limit,

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N) \qquad |a_n - R| < \varepsilon.$$

In particular, we can choose $\varepsilon < 1 - R$, and then for every $n > N$, $|a_n|^{1/n} < R + \varepsilon =: R' < 1$. But then $|a_n| < R'^n$, so $|a_n|$ is bounded above by a geometric series with common ratio smaller than 1, and hence $\sum_n |a_n|$ converges by the Comparison Test. Thus $\sum_n a_n$ converges absolutely.

Divergence Suppose that $R > 1$. Then again by the definition of limit, there is always $N \in \mathbb{N}$ so that $|a_n|^{1/n} > R - \varepsilon$ for $n > N$. Choosing $\varepsilon < R - 1$, we find that $|a_n|^{1/n} > R - \varepsilon > 1$, so $|a_n| > 1$ for every $n > N$. But this means that $a_n \not\to 0$, so the series diverges using the Term Test. □

Example ($R = 1$ tells us nothing)
Let $a_n = n^{-p}$ (i.e. consider the p-series). Then

$$|a_n|^{1/n} = (n^{1/n})^{-p},$$

and since $n^{1/n} = \exp(n^{-1} \log n) \to 1$, we find $R = 1$ in the Root Test. But we know that this series converges if $p = 2$ and diverges if $p = 1$, so $R = 1$ cannot distinguish between them.

We now recall the Basic Ratio Test, and demonstrate that the Basic Root Test is stronger.

Theorem 2 (Basic Ratio Test). *Let $\sum_n a_n$ be a complex series. Suppose that $r := \lim_{n\to\infty} |a_{n+1}/a_n|$ exists (note that this requires that beyond a certain n, every term is nonzero so that the ratio is defined). Then*

Convergence *If $r < 1$, $\sum_n a_n$ converges absolutely.*

Divergence *If $r > 1$, $\sum_n a_n$ diverges.*

No conclusion can be drawn if $r = 1$.

Proposition 3. *If the Basic Ratio Test gives a definite conclusion, so does the Basic Root Test.*

Proof. We show that if r exists, then R exists and is equal to r. If r exists, for any $\varepsilon > 0$ there is N so that for any $n > N$,

$$r - \varepsilon < \left| \frac{a_{n+1}}{a_n} \right| < r + \varepsilon.$$

For ε small enough, all three sides are positive, so we can multiply these together for $N \leqslant n \leqslant m - 1$ to find

$$(r - \varepsilon)^{m-N} |a_N| < |a_m| < (r + \varepsilon)^{m-N} |a_N|.$$

[1] A reminder that none of the tests proven in this handout are in the Schedules, so if you want to use them in examinations, you must prove them first. However, as will become apparent, the proofs are quite straightforward.

The function $t \mapsto t^{1/m}$ is increasing, so applying it preserves the inequalities, giving

$$(r-\varepsilon)\left(\frac{|a_N|}{(r-\varepsilon)^N}\right)^{1/m} < |a_m|^{1/m} < (r+\varepsilon)\left(\frac{|a_N|}{(r+\varepsilon)^N}\right)^{1/m}.$$

Since $b^{1/m} \to 1$ for any b, we can choose m large enough that the left-hand side is subsequently larger than $r-2\varepsilon$ and the right-hand side is smaller than $r+2\varepsilon$. Thus we obtain $\left||a_m|^{1/m}-r\right| < 2\varepsilon$. Since this is true for any $\varepsilon > 0$, it follows that $|a_m|^{1/m} \to r$ as well. □

Of course, this means that we are not surprised that the Root Test does not give a result for the p-series, since the limit is the same as the Ratio Test's limit.

Example (The Root Test can improve on the Ratio Test)

A slightly more artificial example is $\sum_n \alpha^{n+(-1)^n}$. Then the ratio of successive terms is $|\alpha|^{1-2(-1)^n}$, which alternates between $|\alpha|^3$ and $1/|\alpha|$, so we learn nothing from the ratio test since the limit doesn't exist (unless $\alpha = 1$, in which case we already know the answer). On the other hand,

$$|a_n|^{1/n} = |\alpha|^{1-(-1)^n/n} \to |\alpha| \quad \text{as} \quad n \to \infty,$$

so we see that the series converges if $|\alpha| < 1$.

* Improved Root and Ratio Tests

We now give a useful extension of this test, which is much stronger: it disposes of the need for $|a_n|^{1/n}$ to have a limit, and sometimes enables us to reach a conclusion even when the limit is 1.

Theorem 4 (Better Root Test). *Let $\sum_n a_n$ be a complex series. Then*

Convergence *If there is $A < 1$ and $N \in \mathbb{N}$ so that $|a_n|^{1/n} < A$ for all $n > N$, $\sum_n a_n$ converges absolutely.*

Divergence *If $|a_n|^{1/n} \geqslant 1$ for infinitely many n, $\sum_n a_n$ diverges.*

The proof is very similar to before. If anything, removing the limits makes the proof easier:

Proof. **Convergence** We have $|a_n| < A^n$ for $n > N$, and since $A < 1$, $\sum_n A^n$ converges, so by the Comparison Test, so does $\sum_n |a_n|$.

Divergence If $|a_n|^{1/n} \geqslant 1$ for infinitely many n, $|a_n| \geqslant 1$ for infinitely many n, so $\sum_n a_n$ diverges by the Term Test. □

Remark 5. A slightly weaker version of this can be expressed using the quantity

$$L = \limsup_{n\to\infty} |a_n|^{1/n} :$$

if $L < 1$, the series converges, while if $L > 1$, the series diverges. While this looks simpler, it's actually worse, since it does not give a conclusion if $L = 1$ but infinitely many terms are at least 1: this occurs even in the really simple example $a_n = n$.

A natural question is whether the Ratio Test has a similar improvement. The answer is yes, but it looks a bit different.

Theorem 6 (Better Ratio Test). *Let $\sum_n a_n$ be a complex series. Then*

Convergence *If there is $A < 1$ and $N \in \mathbb{N}$ so that $|a_{n+1}/a_n| < A$ for all $n > N$, $\sum_n a_n$ converges absolutely.*

Divergence *If there is $N \in \mathbb{N}$ so that $|a_{n+1}/a_n| \geqslant 1$ for all $n > N$, $\sum_n a_n$ diverges.*

Proof. **Convergence** We have $|a_{n+1}/a_n| < A$ for $n > N$, and multiplying together for $N \leqslant n \leqslant m-1$ gives $|a_m| < A^m(a_N/A^N)$, and the right-hand side form the terms of a geometric series with common ratio $A < 1$, so $\sum_n A^n$ converges, so by the Comparison Test, so does $\sum_n |a_n|$.

Divergence If $|a_{n+1}/a_n| \geqslant 1$ for $n > N$, $|a_{n+1}| \geqslant |a_n| \geqslant \cdots \geqslant |a_N|$ so $a_n \nrightarrow 0$ and $\sum_n a_n$ diverges by the Term Test. □

Remark 7. Notice that the divergence condition is *not* the same as that in the Better Root Test, because the ratios can fluctuate while overall the terms tend to 0: see the previous Example. The conditions can alternatively be written as

$$\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$$

for convergence and

$$\liminf_{n\to\infty} |a_{n+1}/a_n| > 1$$

for divergence, which emphasises further the difference between the tests.