Asymptotic Methods

A Course in Part II of the Mathematical Tripos, University of Cambridge, England

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Contents

0. Schedule

ASYMPTOTIC METHODS (D) **16 lectures, Lent term**

Either Complex Methods or Complex Analysis is essential, Part II Further Complex Methods is useful.

Asymptotic expansions
Definition (Poincare) of $\phi(z) \sim \sum a_n z^{-n}$; examples; elementary properties; uniqueness; Stokes's phenomenon. [4]

Asymptotics behaviour of functions defined by integrals
Integration by parts. Watson's lemma and Laplace's method. Riemann–Lebesgue lemma and method of stationary phase. The method of steepest descent (including derivation of higher order terms). Airy function, *and application to wave theory of a rainbow*. [7]

Asymptotic behaviour of solutions of differential equations
Asymptotic solution of second-order linear differential equations, including Liouville–Green functions (proof that they are asymptotic not required) and WKBJ with the quantum harmonic oscillator as an example. [4]

Recent developments
Further discussion of Stokes' phenomenon. *Asymptotics 'beyond all orders'*. [1]

Appropriate books

† M.J. Ablowitz and A.S. Fokas *Complex Variables: Introduction and Applications.* CUP 2003
J.D. Murray *Asymptotic Analysis.* Springer 1984
A. Erdelyi *Asymptotic Expansions.* Dover 1956
P.D. Miller *Applied Asymptotic Analysis.* American Math. Soc. 2006
F.W.J. Olver *Asymptotics and Special Functions.* A K Peters 1997
† C.M. Bender and S.A. Orszag *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory.* Springer 1999

1. Introduction

Trying to solve [differential] equations is a youthful aberration that you will soon grow out of.

'A well-known mathematician/physicist'[1]

You may have been fortunate enough in your mathematical career thus far to have always been able to solve every differential equation you have encountered analytically, or to compute in closed form the value of every integral you meet. If that is the case, then you have been lied to: this basically never happens in mathematics.[2]

Suppose, therefore, that we cannot evaluate a function exactly. How can we approximate it by functions that we better understand? The first result you know like this is, of course, Taylor's theorem:

Theorem 1. *Suppose that $f(x)$ is N-times continuously differentiable on $[a, b]$. Then*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x), \tag{1.1}$$

where $R_N^{(n)}(x) \to 0$ for each $n \in \{0, 1, \dots, N\}$.

As you know, this is an extremely useful result, but is limited in application: it works on finite (sometimes very small) intervals, and only produces polynomials.

In this course, we are interested in *asymptotic expansions*, which approximate a function in a way similar to Taylor series. However, these expansions often have a fundamentally peculiar property: taking infinitely many terms, the series may converge for no value of x, but taking finitely many (often as far as the smallest term) provides an excellent approximation of the function in a limited region.

For example, consider the integral

$$I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt, \qquad x \geqslant 0; \tag{1.2}$$

this is sometimes called a Steiltjes integral.[3] A couple of facts are obvious here:

1. $I(x)$ will converge for any $x \geqslant 0$, because the integrand is bounded above by e^{-t}.

2. Unless you're very good with special functions, you have no idea what this integral actually evaluates to.[4]

[1] Attributed thus in one of the many repositories of Cambridge mathematical quotations.

[2] Or, indeed, physics. Physicists know they can approximate everything by harmonic oscillators, though.

[3] Not to be confused with the Riemann–Steiltjes and Lebesgue–Steiltjes integrals that you've also never used. Steiltjes is cursed by the inability of any non-native Dutch speaker to pronounce and spell his name either correctly or consistently.

[4] It's a multiple of the incomplete Gamma-function, if you're curious.

So, what do we normally do when we don't know much about a function? Series expand it. We have

$$\frac{1}{1+xt} = \sum_{n=0}^{N} (-1)^n x^n t^n + \frac{(-xt)^{N+1}}{1+xt}, \tag{1.3}$$

so

$$I(x) = \int_0^\infty \left(\sum_{n=0}^{N} (-1)^n x^n t^n + \frac{(-xt)^{N+1}}{1+xt} \right) dt = \sum_{n=0}^{N} (-1)^n n! x^n + (-1)^{N+1} \int_0^\infty \frac{(xt)^{N+1} e^{-t}}{1+xt} \, dt, \tag{1.4}$$

Now, for a while the terms in the sum decrease in value. But eventually, for any particular value of x, the factorial's increase will overtake the decrease in the power, and the terms will increase in absolute value. This means that if we take $N \to \infty$, the series can't possibly converge, for any $x \neq 0$: the radius of convergence is zero.

Now look at the remainder term. We can see that, if x and n are both small enough, the remainder is still small, and hence the sum *will* be a reasonable approximation. Therefore, it seems that the series could be useful: even if taking *all* the terms does not represent a legitimate, finite quantity, taking *some finite number* of terms can produce an excellent approximation.

We will say more about the specific way to choose this in the last section of the course, but for now we give

Rule (Optimal Truncation). To obtain the best approximation to a function value from its divergent series expansion, truncate the series at the smallest term.

Obviously this does depend on the value of x: the approximation is not *uniform*.

In this course, we shall consider a number of ways to obtain these asymptotic expansions, sometimes being able to produce the entire series, and sometimes only being able to find the leading-order term. In many applications, the leading-order term is actually all that is required: you know several examples of this already, such as Stirling's formula,

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}, \quad \text{as } n \to \infty, \tag{1.5}$$

or the Prime Number Theorem,

$$\pi(n) \sim \frac{n}{\log n} \quad \text{as } n \to \infty, \tag{1.6}$$

where $\pi(n)$ counts the number of primes less than n. Both of these have many important applications by themselves, some of which you have already seen in Tripos. There are many other applications in both Mathematics and physics, so this course may serve you well in years to come.[5]

The course is structured around application to specific functions, but we use several pieces of general theory, which we will treat carefully and generally, so that you are capable, if necessary, of using and extending the results yourself.

In the second chapter we give all the definitions that we shall require. In the third, the main section of the course, we study the asymptotics of various very common forms of integral. The fourth chapter changes tack to instead consider asymptotics of the solutions to differential equations, which behave very differently. The last, short, section discusses some general features of asymptotic expansions, such as the optimal truncation rule given above.

[5] Indeed, the author's first published paper used an extension of the results in this course.

2. Asymptotic Expansions

2.1. A reminder of notation

Before we begin, let's have a look at some of the notions of approximation that you already know. Recall the following definitions, which you met in the Differential Equations course:

Definition 2 (O notation). Suppose that f, g are real-valued functions. We say that

$$f(x) = o(g(x)) \quad \text{as } x \to a \tag{2.1}$$

"$f(x)$ is little-oh of $g(x)$ as $x \to a$" if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0. \tag{2.2}$$

Similarly, we say

$$f(x) = O(g(x)) \quad \text{as } x \to a \tag{2.3}$$

"$f(x)$ is big-oh of $g(x)$ as $x \to a$" if there is a nonzero constant c such that

$$f(x) \leqslant cg(x) \quad \text{as } x \to a \tag{2.4}$$

for some finite constant c.

Remark 3. In some cases we shall be interested in limits that only exist from one direction. We shall write $x \uparrow a$ to mean that x increases to a, for example. In the case of $a = +\infty$, we shall just write $x \to +\infty$.

Legal Notice 4. *It is well-known that this notation is dodgy: it messes with things like the symmetry of equality and so on. We shall nevertheless continue this abuse of notation, or we'd end up writing things like*

$$(x+1)^2 - x^2 \in O(x) \quad \textit{as } x \to \infty,$$

which would drive us all to distraction.

These notations are useful, but fundamentally they are concerned with throwing stuff away in taking a limit. In particular, they contain no information on how large the removed terms are, so they tell us little about non-microscopic ranges near the point a. We now introduce a new concept, asymptotic convergence, which is chiefly concerned with the terms that *are* important when x is close to, but not equal to, a.

2.2. Asymptotic equality

Definition 5 (Asymptotic equality). We say that $f(x)$ is asymptotically equal to $g(x)$ as $x \to a$, and write

$$f(x) \sim g(x) \quad \text{as } x \to a, \tag{2.5}$$

if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1. \tag{2.6}$$

This is *the* most important definition in the course, and we shall use it constantly.

Remark 6. Be careful when you see $\sim$ used like this in an equation: in some subjects, such as physics, they have sufficiently poor taste that they use it to mean other things, such as *proportional to* or *looks vaguely similar to.*

Exercise Show that the given definition is equivalent to

$$f(x) \sim g(x) \iff f(x) - g(x) = o(g(x)). \tag{2.7}$$

We collect some basic results about $\sim$ in the next lemma.

Lemma 7. *Asymptotic equality as $x \to a$ has the following properties:*

1. *Linearity: if $f_1 \sim g_1$ and $f_2 \sim g_2$ and α and β are constants, then $\alpha f_1 + \beta f_2 \sim \alpha g_1 + \alpha g_2$.*
2. *Products: if $f_1 \sim g_1$ and $f_2 \sim g_2$, then $f_1 f_2 \sim g_1 g_2$.*
3. - *Reflexivity: $f \sim f$,*
   - *Symmetry: $f \sim g \iff g \sim f$,*
   - *Transitivity: if $f \sim g$ and $g \sim h$, then $f \sim h$.*

   Therefore, for those of you who care, $\sim$ is an equivalence relation on the set of (for example) continuous functions that do not vanish on an open interval containing a.

2.3. Asymptotic expansion

We shall define what it means for a function to have an *asymptotic expansion.*[1] The formal notion looks quite abstract, but we shall see that making a fairly general definition will help later.[2]

Definition 8. An *asymptotic sequence* at a is a sequence of functions $(\phi_k)_{k=1}^{\infty}$ such that $\phi_{k+1}(x) = o(\phi_k(x))$ as $x \to a$, for every k.

If a is finite, an obvious example is the powers of $(x - a)$, since

$$(x-a)^{k+1} = o((x-a)^k) \quad \text{as} \quad x \to a. \tag{2.8}$$

[1]This was first considered formally by Poincaré (see [8]).
[2]In particular, it allows us to consider a much broader class of expansions than, say, Taylor series-type expansions.

We can start this sequence at $k < 0$ if we so desire. Other possibilities include a sequence of fractional powers, $x^{\alpha+k\beta}$, or $\log|x|, \log|\log|x||, \ldots$, or

$$\left(\frac{x^k}{1+x}\right)_{k=K}^{\infty}, \tag{2.9}$$

and so on.

If $a = \infty$, then

$$x^n, x^{n-1}, \ldots, x, 1, \frac{1}{x}, \frac{1}{x^2}, \ldots \tag{2.10}$$

is an example, as are

$$\frac{1}{x}, \frac{1}{x(x+1)}, \frac{1}{x(x+1)(x+2)}, \ldots, \tag{2.11}$$

and

$$e^{\alpha_1 x}, e^{\alpha_2 x}, e^{\alpha_3 x}, \ldots \tag{2.12}$$

for any strictly decreasing sequence $(\alpha_k)_{k=1}^{\infty}$. Also valid is something as irregular as

$$2x, \frac{1}{x}, \frac{1}{(1+x)^2}, \frac{1}{x^{5/2}}, \ldots, \tag{2.13}$$

provided subsequent terms also follow the definition.

Having looked at plenty of examples, we now reach the point of the section.

Definition 9. Let f be a function on a domain with a as a limit point (that is, either a is inside the domain, or is a boundary point). f is said to have an *N-term asymptotic expansion at a* in terms of (ϕ_k) if there are constants $(a_k)_{k=1}^{N}$ such that

$$f(z) = \sum_{k=1}^{N} a_k \phi_k(z) + o(\phi_N(z)) \quad \text{as} \quad x \to a. \tag{2.14}$$

If this relationship holds for any positive integer N, the series $\sum_{k=1}^{N} a_k \phi_k(z)$ is called an *asymptotic expansion of f at a*, and we write

$$f(z) \sim \sum_{k=1}^{\infty} a_k \phi_k(z) \quad \text{as} \quad x \to a. \tag{2.15}$$

This is one of those definitions that looks completely useless: it gives us no way to find an asymptotic expansion! Fear not: the rest of the course will teach some heuristics for computing commonly occuring expansions. However, first let's check some properties of this definition.

Remark 10.

1. An asymptotic expansion may not converge, for any value of the variable z (we saw an example of this in the Introduction).

2. Obviously this definition is linear, so if f, g have asymptotic expansions

$$f(z) \sim \sum_{k=1}^{\infty} a_k \phi_k(z) \quad \text{as} \quad x \to a, \tag{2.16}$$

$$g(z) \sim \sum_{k=1}^{\infty} b_k \phi_k(z) \quad \text{as} \quad x \to a, \tag{2.17}$$

then $\alpha f(z) + \beta g(z)$ has asymptotic expansion

$$\alpha f(z) + \beta g(z) \sim \sum_{k=1}^{\infty} (\alpha a_k + \beta b_k)\phi_k(z) \quad \text{as} \quad x \to a, \tag{2.18}$$

for any constants α, β.

3. A function has different asymptotic expansions with respect to different asymptotic sequences: for example, let $f(z) = 1 + z^{-1}$. Then with respect to the sequence $(z^{-k})_{k=0}^{\infty}$ at ∞,

$$f(z) \sim 1 \cdot 1 + 1 \cdot \frac{1}{z} + 0 \cdot \frac{1}{z^2} + \cdots, \tag{2.19}$$

whereas if we instead consider the sequence $(1+z)^{-k}$, we have that

$$1 + \frac{1}{z} = \frac{1+z}{(1+z)-1} = \left(1 - \frac{1}{1+z}\right)^{-1}, \tag{2.20}$$

so with the second sequence,

$$1 + \frac{1}{z} \sim \sum_{k=0}^{\infty} \frac{1}{(1+z)^k}. \tag{2.21}$$

4. An asymptotic expansion does not uniquely determine a function: for example, as $x \uparrow \infty$, $(1+x)^{-1}$ and $(1+e^{-x})(1+x)^{-1}$ both have asymptotic expansion $\sum_{k=0}^{\infty} (-1)^k z^{-(k+1)}$.

Proposition 11. *Given an f and an asymptotic sequence (ϕ_k) at a, if f has an asymptotic expansion*

$$f(z) \sim \sum_{k=1}^{\infty} a_k \phi_k(z) \quad \textit{as} \quad x \to a, \tag{2.22}$$

this expansion is unique.

Proof. We may proceed by an inductive argument. We are looking for an expansion of the form

$$f(z) = \sum_{k=1}^{N-1} a_k \phi_k(z) + a_N \phi_N(z) + R_N(z), \tag{2.23}$$

where $R_N(z) = o(\phi_N(z))$ as $z \to a$. Then

$$\frac{f(z) - \sum_{k=1}^{N-1} a_k \phi_k(z)}{\phi_N(z)} = a_N + \frac{R_N(z)}{\phi_N(z)}, \tag{2.24}$$

and since the latter term on the right $\to 0$ as $z \to a$, we have a unique determination for a_N:

$$a_N = \lim_{z \to a} \frac{f(z) - \sum_{k=1}^{N-1} a_k \phi_k(z)}{\phi_N(z)}. \tag{2.25}$$

(Note that the proof shows that limit must exist, from the definition of an asymptotic series, and hence the definition of R_N.) This works for any N, including the case $N = 1$, where the limit is just $f(z)/\phi_1(z)$. □

In this course, we are particularly concerned with asymptotic expansions of the form

$$f(z) \sim \sum_{k=0}^{\infty} \frac{a_k}{z^k}. \tag{2.26}$$

2.4. Stokes's Phenomenon

Life is not so simple as we have so far pretended: a function can have different behaviour as it approaches a point from different directions.[3] We should expect this to be reflected in any asymptotic expansions we try to make. Let's look at an example:

Example 1. Suppose $f(z) = \sin z$. Determine asymptotic approximations of $f(z)$ as $|z| \to \infty$.

Right, an important thing to note about the definition of asymptotic is that if $f(z) \sim g(z)$ as $z \to a$, and $f(z)$ has an infinite sequence of zeros $z_n \to a$, then $g(z)$ must also have the same zeros. (Otherwise $f(z_n) - g(z_n) = -g(z_n)$, which is obviously not $o(g(z_n))$. You can also look at the limit explicitly to see this. Therefore if z is real, the only sensible asymptotic is $\sin z$. Very enlightening(!).

Right, now what about complex z? Since we're looking at $|z| \to \infty$, it seems sensible to expand in polar coordinates:

$$\sin(re^{i\theta}) = \frac{1}{2i}(e^{-r\sin\theta}e^{ir\cos\theta} - e^{r\sin\theta}e^{-ir\cos\theta}). \tag{2.27}$$

Suppose that $\sin\theta > 0$ (so z has positive imaginary part). Then $e^{-r\sin\theta}e^{ir\cos\theta}$ is bounded by $e^{-r\sin\theta}$, which is obviously $o(1)$ as $r \to \infty$. On the other hand, $e^{r\sin\theta}$ blows up like $e^{\alpha r}$ for some $\alpha > 0$. Therefore the second term dominates, and we have the asymptotic approximation

$$\sin z \sim -\frac{1}{2i}e^{-iz} \tag{2.28}$$

if the argument of z is between 0 and π, but is not allowed to approach 0 or π in some funny way that makes z approach the real line as an asymptote. A simple way to write this is

$$0 + \varepsilon < \arg z < \pi - \varepsilon, \tag{2.29}$$

for a fixed $\varepsilon > 0$. In exactly the same way, we can show that

$$\sin z \sim \frac{1}{2i}e^{iz} \tag{2.30}$$

as $|z| \to \infty$, for $-\pi + \varepsilon < \arg z < 0 - \varepsilon$.

Therefore for large $|z|$, $\sin z$ has different asymptotic approximations in different parts of the complex plane. This is an example of the *Stokes phenomenon*: an analytic function can have different asymptotic approximations in different parts of the plane.[4] In general, for a large-z asymptotic expansion, there are different dominant terms in different sectors of the plane, which change over on curves where the magnitude of the respective asymptotic basis functions become comparable (in our example, this is the real axis, where e^{iz} and e^{-iz} have similar magnitude). We call these lines, where the dominant and recessive terms swap rôles, Stokes Lines.[5].

Remark 12. You can say the same for something like $x+e^x$ on $\mathbb{R}$, of course, but here the phenomenon is just different behaviour for large and large negative x. The business with approximations changing over on Stokes lines, however, can only be a complex phenomenon, since we have a continuous variation in the directions in which z can increase.

We will go into more detail later on, when we discuss the Airy function and its connexion to the WKB[...] method.

[3]But you knew that, of course.
[4]And for once, it seems it actually was Stokes that discovered it.
[5]Others may also call them Anti-Stokes lines, depending on convention; the literature is not consistent

3. Approximations of Integrals

In this chapter, the main section of the course, we will consider how to approximate some of the common integrals that it is impossible to evaluate exactly. We begin with some basic examples that only require cheap tricks to evaluate. We then move on to an extremely general (and, indeed, surprising) asymptotic result, Watson's Lemma, which is capable of giving us the entire asymptotic expansion of many integrals, and only requires that the functions involved satisfy quite weak conditions.

After that, we discuss integrals involving $e^{\phi(x)}$, which are ubiquitous in applications. We begin with Laplace's Method, its imaginary friend the Method of Stationary Phase, and then we shall extend both of these results considerably into what is known as the Method of Steepest Descent.

Finally, in preparation for the next chapter, we apply the theory to this course's pet function: the Airy function; this function will prove to be key in making approximations to the solutions of differential equations, the subject of the next chapter.

3.1. Integration by parts

Our first technique is something you all know very well, and have since you were at school:

Theorem 13. *Let f, g be continuously differentiable, with integrable derivatives on the open interval (a, b). Then*

$$\int_a^b f'(t)g(t)\,dt = [f(t)g(t)]_a^b - \int_a^b f(t)g'(t)\,dt, \tag{3.1}$$

where the term in square brackets is treated as $\lim_{t\uparrow b} - \lim_{t\downarrow a}$.

I sense you are unimpressed. *Patience, my friend.*[1] But it's possible that one product is smaller than the other, so integrating by parts may make the remainder integral relatively small. Let's have an

Example 2. Suppose $x \gg 1$. Let

$$F(x) = \int_x^\infty e^{-t^2/2}\,dt; \tag{3.2}$$

which we know is a useful function, but with no closed form for the antiderivative.[2]

We shall find the asymptotic expansion of this integral by integrating by parts. Write the integrand as $\frac{1}{t}te^{-t^2/2}$. Then we can integrate the second term exactly:

$$F(x) = \left[-\frac{1}{t}e^{-t^2/2}\right]_x^\infty - \int_x^\infty \frac{1}{t^2}e^{-t^2/2}\,dt \tag{3.3}$$

$$= 0 + \frac{e^{-x^2/2}}{x} - \int_x^\infty \frac{e^{-t^2/2}}{t^2}\,dt \tag{3.4}$$

[1] ~~*In time, he will seek you out, and when he does, you must bring him before me.*~~

[2] You should recognise this function as related to the cumulative distribution function of a standard normal distribution: if $\Phi(x)$ is the CDF of the normal distribution, $\Phi(x) = 1 - F(x)/\sqrt{2\pi}$.

Okay, have we actually done anything here? Fundamentally what we have to do now is *check if the error term is small.* After all, it's no good carrying out this procedure and ending up with an integral that we can't do that is larger than the one we can! In this case, we have done the right thing:

$$\left| F(x) - \frac{e^{-x^2/2}}{x} \right| = \int_x^\infty \frac{e^{-t^2/2}}{t^2}\,dt, \tag{3.5}$$

since the integrand is positive. To show that we have an asymptotic expansion, we have to show that the right-hand side is $o(e^{-x^2/2}/x)$. We can use a similar trick to before, but this time, notice that since in the integral $t > x$, we have $\frac{1}{t} < \frac{1}{x}$. Hence:

$$\begin{aligned} \int_x^\infty \frac{e^{-t^2/2}}{t^2}\,dt &= \int_x^\infty \frac{1}{t^3} t e^{-t^2/2}\,dt && (3.6)\\ &\leqslant \frac{1}{x^3}\int_x t e^{-t^2/2}\,dt && (3.7)\\ &= \frac{1}{x^3} e^{-x^2/2}. && (3.8) \end{aligned}$$

Therefore,

$$\frac{x^2}{e^{-x^2/2}} \int_x^\infty \frac{e^{-t^2/2}}{t^2}\,dt \leqslant \frac{1}{x} \to 0 \tag{3.9}$$

as $x \to +\infty$, and hence

$$F(x) \sim \frac{e^{-x^2/2}}{x} \quad \text{as} \quad x \to +\infty. \tag{3.10}$$

Indeed, why stop at one integration by parts? We can repeat this as many times as we like, to obtain

$$\int_x^\infty e^{-t^2/2}\,dt \sim \frac{e^{-x^2/2}}{x} \sum_{n=0}^\infty (-1)^n \frac{(2n-1)!!}{x^{2n}}, \tag{3.11}$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 3 \cdot 1$ (the double factorial).

Exercise Prove this. [Hint: consider using induction, after determining the form of the remainder term.]

3.1.1. A quick skim through the Gamma-function

At some point in your mathematical career thus far, you should have encountered the generalisation of the factorial known as the Gamma-function. If not, here I shall do a quick runthrough of the properties we shall need in the rest of the course. Proofs will be left as exercises or just skipped. (And whether I even mention this section in the lectures is up in the air: we'll see.)

As is typical in mathematics, we define the thing so that it has the property that we want it to have.

Definition 14. Let $\Re z > 0$. The Gamma-function is the complex-valued function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{3.12}$$

Lemma 15 (Basic properties).

1. *$\Gamma(z)$ is an analytic function of z for $\Re z > 0$.*
2. *Functional equation:*

$$z\Gamma(z) = \Gamma(z+1) \tag{3.13}$$

3. *If n is a non-negative integer,*

$$\Gamma(n+1) = n!. \tag{3.14}$$

4. *$\Gamma(z)$ extends uniquely to a meromorphic function on $\mathbb{C}$, with poles at non-positive integers.*
5. *Suppose $x > 0$. Then*

$$\int_0^\infty t^{z-1} e^{-xt}\, dt = \frac{\Gamma(z)}{x^z}. \tag{3.15}$$

6. *Reflexion formula:*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{3.16}$$

7. *Duplication formula:*

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \tag{3.17}$$

8. *In particular, let $a > 0$. Then*

$$\int_{-\infty}^\infty e^{-a^2x^2}\, dx = \int_0^\infty y^{-1/2} e^{-a^2 y}\, dy = \frac{\Gamma(1/2)}{a^{2/2}} = \frac{\sqrt{\pi}}{a}, \tag{3.18}$$

by putting $z = 1/2$ in the refection formula or the duplication formula.

These should all be proved in the Further Complex Methods course. Perhaps not all of these will be required, but you should know it anyway, 'cos it's cool.[3]

[3] On the other hand, I will spare you a dissertation on Elliptic Functions and Their Many Ultra-Cool Properties. Nor shall I go into the many other formulae for the Gamma-function that we definitely shall not use.

3.2. Watson's lemma

'Pon my word, Watson, you are coming along wonderfully. You have really done very well indeed.

Sherlock Holmes, *A Case of Identity*
Sir Arthur Conan Doyle[4]

Not, in fact, John H. Watson, M.D., Late of the Army Medical Department, but G.N. Watson (1886–1965), famous primarily for four things: going to Trinity, his collaboration on the second and subsequent editions of Whittaker and Watson,[5] his book on Bessel functions,[6] and this lemma, first published in 1918 in a paper on parabolic cylinder functions.[7] To characterise its importance within the course, we shall elevate it to

Theorem 16. *Let $0 < T \leqslant \infty$. Suppose that $f(t)$ has an asymptotic expansion at zero of the form*

$$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n}, \quad t \downarrow 0 \tag{3.19}$$

with $\alpha > -1$, and that in addition, either

1. $|f(t)| < Ke^{bt}$ *for every* $t > 0$, *or*
2. $\int_0^T |f(t)| \, dt < \infty$.

Then

$$F(x) := \int_0^T e^{-xt} f(t) \, dt \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}}, \quad x > 0, x \to +\infty. \tag{3.20}$$

Now that's nice! If we can find an asymptotic expansion for $f(t)$ for small t, we automatically get one for the integral $F(x)$. Note also that, by the definition of asymptotic expansion, we can translate a finite asymptotic expansion of $f(t)$ into one of the same length for $F(x)$. Without further ado, the

Proof. Since we have an asymptotic expansion at $t = 0$, it is strongly suggested that we partition the integral into two: one over a small interval around this point, added to one over the rest of the interval. We can then (hopefully) show that the latter part is small.[8] Therefore write

$$F(x) = \int_0^{\varepsilon} e^{-xt} f(t) \, dt + \int_{\varepsilon}^{T} e^{-xt} f(t) \, dt, \tag{3.21}$$

where $\varepsilon > 0$ is small enough that the asymptotic expansion applies. Next, we need to bring in the series in the theorem. A calculation (or a result in the previous section) shows that

$$\int_0^{\infty} e^{-xt} t^{\lambda} \, dt = \frac{\Gamma(1+\lambda)}{x^{\lambda+1}}, \tag{3.22}$$

[4] Shamelessly appropriated from Bender and Orszag's book [1].

[5] [13], the obsolescently-titled *A Course of Modern Analysis*; presumably we are now well into Postmodern Analysis and subsequent degenerations.

[6] [11], the definitive book on the subject.

[7] They're essentially generalisations of solutions to the Hermite differential equation, since you ask. [12], p. 133.

[8] This idea will return repeatedly during this course: the fundamental idea of splitting off the dominant interval is a large part of the theory here.

which hopefully explains where the asymptotic series comes from. Therefore, let us look at the error of approximation:

$$R_N(x) = \left| F(x) - \sum_{n=0}^{N} a_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \right|. \tag{3.23}$$

To show that we have an asymptotic expansion, we need to show that $R_N(x) = o(x^{-\alpha-\beta N-1})$ as $x \uparrow \infty$. First, apply our Gamma-function identity to the sum, and separate off the interval that we suspect is dominant:

$$\begin{aligned} R_N(x) &= \int_0^\varepsilon e^{-xt} f(t)\,dt + \int_\varepsilon^T e^{-xt} f(t)\,dt - \int_0^\infty e^{-xt} \sum_{n=0}^{N} a_n t^{\alpha+\beta n}\,dt && (3.24)\\ &= \int_0^\varepsilon e^{-xt} \left(f(t) - \sum_{n=0}^{N} a_n t^{\alpha+\beta n} \right) dt + \int_\varepsilon^T e^{-xt} f(t)\,dt - \int_\varepsilon^\infty e^{-xt} \sum_{n=0}^{N} a_n t^{\alpha+\beta n}\,dt && (3.25) \end{aligned}$$

Now, to look at the size of this, we can apply the triangle inequality:

$$\begin{aligned} |R_N(x)| &= \left| \int_0^\varepsilon e^{-xt} \left(f(t) - \sum_{n=0}^{N} a_n t^{\alpha+\beta n} \right) dt + \int_\varepsilon^T e^{-xt} f(t)\,dt - \int_\varepsilon^\infty e^{-xt} \sum_{n=0}^{N} a_n t^{\alpha+\beta n}\,dt \right| && (3.26)\\ &\leqslant \left| \int_0^\varepsilon e^{-xt} \left(f(t) - \sum_{n=0}^{N} a_n t^{\alpha+\beta n} \right) dt \right| + \left| \int_\varepsilon^T e^{-xt} f(t)\,dt \right| + \left| \int_\varepsilon^\infty e^{-xt} \sum_{n=0}^{N} a_n t^{\alpha+\beta n}\,dt \right| && (3.27)\\ &\leqslant \int_0^\varepsilon e^{-xt} \left| f(t) - \sum_{n=0}^{N} a_n t^{\alpha+\beta n} \right| dt + \int_\varepsilon^T e^{-xt} \left| f(t) \right| dt + \left| \int_\varepsilon^\infty e^{-xt} \sum_{n=0}^{N} a_n t^{\alpha+\beta n}\,dt \right| && (3.28)\\ &=: R_1 + R_2 + R_3 && (3.29) \end{aligned}$$

The result now rests on the bounding of these three integrals. The last one is straightforward:

$$R_3 = e^{-\varepsilon x} \left| \int_\varepsilon^\infty e^{-x(t-\varepsilon)} \sum_{n=0}^{N} a_n t^n\,dt \right| = \frac{e^{-\varepsilon x}}{x} \left| \int_0^\infty e^{-u} \sum_{n=0}^{N} a_n \left(\varepsilon + \frac{u}{x} \right)^n du \right| < \frac{e^{-\varepsilon x}}{x} K_3, \tag{3.30}$$

having changed variables to $u = x(t-\varepsilon)$. The remaining integral can be easily bounded, for example by applying the binomial theorem and the integral definition of the Gamma-function.[9] Hence R_3 is exponentially suppressed.

Fo the first term, we can choose ε so that the integrand is bounded above by $K_1 t^{\alpha+\beta(N+1))}$ (this is just a restatement of the asymptotic expansion property in terms of "big-O" rather than "little-O"). Therefore

$$\begin{aligned} R_1 &< K_1 \int_0^\varepsilon e^{-xt} t^{\alpha+\beta(N+1)}\,dt = K_1 \int_0^\infty e^{-xt} t^{\alpha+\beta(N+1)}\,dt = K_1 \frac{\Gamma(\alpha+\beta(N+1)+1)}{x^{\alpha+\beta N+1} x^\beta} && (3.31)\\ &= o(x^{-\alpha-\beta N-1}) && (3.32) \end{aligned}$$

as $x \uparrow \infty$.

The second term is dealt with in different ways depending on which of hypotheses (1) or (2) the function satisfies.

[9] One can also integrate by parts in much the same way as in the previous section, but this way is both quicker and sufficient.

1. This term is bounded above as follows:

$$R_2 < \int_\varepsilon^T K_2 e^{(b-x)t}\,dt \leqslant \int_\varepsilon^\infty K_2 e^{(b-x)t}\,dt = K_2 \frac{e^{(b-x)\varepsilon}}{x-b} = O(x^{-1}e^{-\varepsilon x}) \tag{3.33}$$

   as $x \uparrow \infty$.

2. In this case, note that

$$R_2 < e^{-\varepsilon x} \int_0^T |f(t)|\,dt; \tag{3.34}$$

   the integral is finite by assumption, so this term is also exponentially supressed, $O(e^{-\varepsilon x})$.

In either case, we have shown that all three terms are $o(x^{-\alpha-\beta N-1})$. This applies for any $N \geqslant 0$, and so the expansion is asymptotic. □

Now, this proof is neat-looking, if you're into this sort of thing, but perhaps a bit unsatisfying: where did the asymptotic expansion come from? Here is a breakdown of what's going on in this proof:

1. We have an expansion of the function near the origin. We don't know how far away this is valid to, so split the integral into a small interval on which we believe the asymptotic expansion of $f(t)$ holds, and the rest:

$$\int_0^\infty f(t)e^{-xt}\,dt = \int_0^\varepsilon f(t)e^{-xt}\,dt + \int_\varepsilon^T f(t)e^{-xt}\,dt. \tag{3.35}$$

2. Now, e^{-xt} becomes small very quickly as x becomes large, so we expect the main term to be the first integral. The first remainder term (R_2 in the proof above) we deal with by bounding it above, using one of the two conditions in the theorem.

3. Replacing $f(t)$ by its series expansion, we have

$$\int_0^\varepsilon \left(\sum_{n=0}^N a_n t^{\alpha+\beta n} e^{-xt} + e^{-xt} R_N(t) \right) dt, \tag{3.36}$$

   where $R_N(t)$ is the remainder term in the $f(t)$ asymptotic expansion. Once again, the main term should be the first term in the equation. In particular, we know that $R_N(t) = o(t^{\alpha+\beta N})$ in order for the series expansion around $t = 0$ to be asymptotic. We can eventually use this to show that the second remainder term (R_1 in the proof above) is smaller than the last term we are retaining, and so the series is asymptotic.

4. The last problem is to change the integral of the sum into an integral we can actually do. To do this, we add and subtract an integral that we think will be small: namely, the missing $\int_\varepsilon^\infty$ of the approximation:

$$\int_0^\infty \sum_{n=0}^N a_n t^{\alpha+\beta n} e^{-xt}\,dt - \int_\varepsilon^\infty \sum_{n=0}^N a_n t^{\alpha+\beta n} e^{-xt}\,dt. \tag{3.37}$$

   Now we can do the first integral, which gives the asymptotic expansion sum that we want, and bound the second integral to show that the last error we introduced (R_3 above) really was small.

So, we chopped the integral into a small interval where we think the function is large, and a large interval which we thought was unimportant. We then made an approximation on the small part, and showed that the error of the approximation would small enough to ignore. We then extend the interval of integration back out, to make the integral of the approximation doable analytically. This introduces an error, which we also show is small.[10]

Example 3 (Silly example). Let's start with an example we can do exactly to check that we know what we're doing. Consider

$$I(x) = \int_0^\infty e^{-xt} \sin t \, dt. \tag{3.38}$$

We know that

$$\sin t = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} t^{2n+1}, \tag{3.39}$$

and $\sin t$ is bounded on $(0, \infty)$, so we can apply Watson's lemma:

$$I(x) \sim \sum_{n=0}^\infty (-1)^n \frac{\Gamma(2n+2)}{(2n+1)!} \frac{1}{x^{2n+2}} = \frac{1}{x^2} \sum_{n=0}^\infty \frac{(-1)^n}{x^{2n}}. \tag{3.40}$$

But of course we know that

$$I(x) = \frac{1}{1+x^2} = \frac{1}{x^2}(1+x^{-2})^{-1}, \tag{3.41}$$

which has exactly the same series expansion for large x. So it works!

Example 4 (Confluent hypergeometric function). One solution to the *confluent hypergeometric differential equation*,

$$xy'' + (b-x)y' - ay = 0, \tag{3.42}$$

is given by

$$M(a,b,x) = \frac{e^x}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{-xt}(1-t)^{a-1} t^{b-a-1} \, dt, \quad \Re(b) > \Re(a) > 0 \tag{3.43}$$

We shall find the large-x asymptotics. We know immediately that M is in a form appropriate for Watson's lemma: near $t=0$ we have the Taylor series expansion

$$t^{b-a-1}(1-t)^{a-1} = \sum_{n=0}^\infty (-1)^n \frac{\Gamma(a)}{\Gamma(a-n)n!} t^{b-a+n-1}, \tag{3.44}$$

and the function which is $t^{b-a-1}(1-t)^{a-1}$ on $(0,1)$ and 0 otherwise has finite integral with the given conditions on a and b, so we find

$$M(a,b,x) \sim \frac{e^x}{\Gamma(a)} \sum_{n=0}^\infty (-1)^n \frac{\Gamma(b-a+n)}{\Gamma(a)\Gamma(b-a)\Gamma(a-n)n!} \frac{1}{x^{b-a+n}}. \tag{3.45}$$

In particular, the leading order term is $e^x x^{a-b}/\Gamma(a)$.

[10] This is a very common approach to producing asymptotic expansions, and we will do it again in the next few sections.

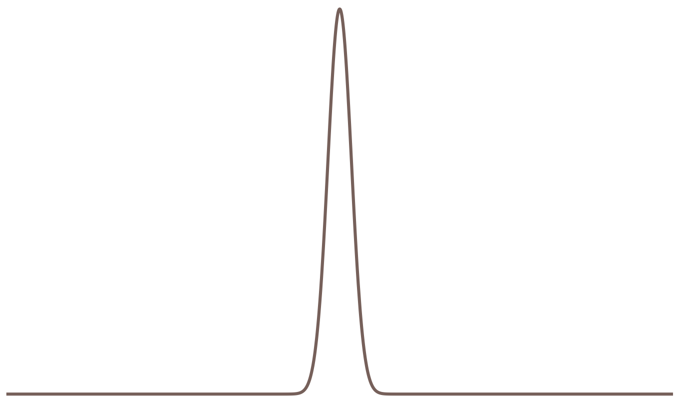

Figure 3.1.: Plot of $e^{x\phi(t)}$ for a simple ϕ. For large x, by far the largest contribution to the integral (3.47) is from a small interval around the maximum of ϕ.

Exercise A linearly independent solution to the confluent hypergeometric equation is given by

$$U(a,b,x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{b-a-1} \, dt. \tag{3.46}$$

Find the asymptotic expansion of $U(a,b,x)$ for large x. Discuss the behaviour of the Wronskian for large-x.

3.3. Exponentially Decaying Integrals: Laplace's Method

Watson's Lemma deals with functions where an endpoint contributes most: normally a change of variables can place the integrand in the appropriate form.

Suppose now that the integrand has a maximum at an interior point. Specifically, we shall consider integrals of the from

$$F(x) := \int_a^b f(t) e^{x\phi(t)} \, dt, \tag{3.47}$$

for x large. Because the exponential grows so rapidly, it is evident that the contribution from near points where $\phi(t)$ has a maximum will dominate the integral (provided that f is not zero nearby, which we suppose occurs by the Principle of Niceness). Therefore, let's look at what the integral looks like near this maximum.

To do this, of course, we expand the function $\phi(t)$ in a Taylor series:

$$\phi(t) = \phi(c) + \phi'(c)(t-c) + \frac{\phi''(c)}{2}(t-c)^2 + O((t-c)^3). \tag{3.48}$$

Suppose that c is a maximum. Then we have $\phi'(c) = 0$ and $\phi''(c) \leqslant 0$. Usually, we have $\phi''(c) < 0$, so let's look at this case first. We also have $f(t) = f(c) + O(t-c)$.

$$F(x) = \int_a^b f(t) e^{x\phi(t)} \, dt \sim \int_{c-\varepsilon}^{c+\varepsilon} (f(c) + O(t-c)) \exp\left(x\phi(c) + x\frac{1}{2}\phi''(c)(t-c)^2 \right) dt \tag{3.49}$$

$$= f(c) e^{x\phi(c)} \int_{c-\varepsilon}^{c+\varepsilon} e^{x\phi''(c)(t-c)^2/2} \, dt. \tag{3.50}$$

This last integral on the right may look rather Gaussian, but we have the problem that it's over a rather finite-looking interval. Make the substitution $s^2 = -x\phi''(c)(t-c)$, $ds = (-x\phi''(c))^{1/2}$ transforms the integral to

$$F(x) \sim \frac{f(c)e^{x\phi(c)}}{(x\phi''(c))^{1/2}} \int_{-(x\phi''(c))^{1/2}\varepsilon}^{(-x\phi''(c))^{1/2}\varepsilon} e^{-s^2/2}\,dt \tag{3.51}$$

Now the limits have become large, like $x^{1/2}\varepsilon$, and we know from Example 2 that truncating this integral leads to an exponentially small error, so we have

$$F(x) \sim \frac{f(c)e^{x\phi(c)}}{(x\phi''(c))^{1/2}} \int_{-\infty}^{\infty} e^{-s^2/2}\,dt, \tag{3.52}$$

and doing the final integral gives us an exact expression for the leading-order asymptotic:

$$F(x) \sim \left(\frac{2\pi}{-x\phi''(c)}\right)^{1/2} f(c)e^{x\phi(c)} \quad \text{as } x \uparrow \infty. \tag{3.53}$$

Remark 17. In the first part of the course, finding the next term in the asymptotic expansion has mostly been about the same amount of work as finding each previous term. From now on, finding higher-order terms gets exponentially harder[11] as the number of terms increases. (This is actually a typical situation: consider finding the nth derivative of a function, for example. We just happen to have so far played with nice functions most of the time.) This continues all the way into Quantum Field Theory, for those of you who stick with physics that long.

Example 5 (Silly example). Again, let's start with an example we already understand, so that we can check the answer. Consider

$$I(x) = \int_{-\infty}^{\infty} e^{-xt^2} e^{at}\,dt. \tag{3.54}$$

We obviously take $\phi(t) = -t^2$, $f(t) = e^{at}$. The only maximum of ϕ is at $t = 0$, and $\phi''(0) = 2$, so Laplace gives us

$$I(x) \sim \sqrt{\frac{\pi}{x}} \quad \text{as } x \to \infty, \tag{3.55}$$

which agrees with the exact result, $I(x) = e^{-b^2/4x}\sqrt{\pi/x}$.

Example 6. Consider

$$J(x) = \int_0^{\pi} e^{x \sin t}\,dt. \tag{3.56}$$

(this is related to Bessel functions, but is not a particular one). We have

$$\phi(t) = \sin t \tag{3.57}$$
$$\phi'(t) = \cos t \tag{3.58}$$
$$\phi''(t) = -\sin t, \tag{3.59}$$

so we have one turning point, $t = \pi/2$, which is a maximum. We have

$$\phi(\pi/2) = 1, \qquad \phi''(\pi/2) = -1, \tag{3.60}$$

so Laplace tells us that

$$J(x) \sim \sqrt{\frac{\pi}{x}} e^x \quad \text{as } x \to \infty. \tag{3.61}$$

[11]Not a rigorous claim.

Remark 18.
1. Notice that the asymptotic contains $e^{x\phi(c)}$. Then even if the integrand has other local maxima, their contribution is exponentionally smaller than the global maximum at c.
2. Just as with Watson's Lemma, this is a *local* approximation: only the function in the immediate vicinity of c is used: all other contributions are exponentially smaller.
3. Unlike Watson's Lemma, we have only found the first term. Subsequent terms are rather more complicated, as we will discuss below.

3.3.1. Generalisation

Suppose the maximum is instead at an endpoint; without loss of generality, we can take this as a. Suppose first that $\phi'(a) \neq 0$; since the point is supposed to be a maximum, we should then have $\phi'(a) < 0$. Then the next-to-leading-order approximation of $\phi(t)$ as $t \to a$ is

$$\phi(t) \sim \phi(a) + \phi'(c)(t-a). \tag{3.62}$$

As with Watson's lemma, the contribution outside the small region $[a, \varepsilon]$ is exponentially suppressed, and we have

$$\begin{aligned}
F(x) &\sim f(c)e^{x\phi(a)} \int_a^{a+\varepsilon} e^{x\phi'(a)(t-a)}\,dt && (3.63)\\
&= \frac{f(c)e^{x\phi(a)}}{-x\phi'(a)} \int_0^{-x\phi'(a)\varepsilon} e^{-s}\,ds && (3.64)\\
&\sim \frac{f(c)e^{x\phi(a)}}{-x\phi'(a)} \int_0^{\infty} e^{-s}\,ds && (3.65)\\
&= \frac{f(c)e^{x\phi(a)}}{-x\phi'(a)} && (3.66)
\end{aligned}$$

On the other hand, if $\phi'(a) = 0$ and $\phi''(a) < 0$, we can follow the previous proof, and reach

$$F(x) \sim \frac{f(a)e^{x\phi(a)}}{(-x\phi''(x))^{1/2}} \int_0^{(-x\phi(a))^{1/2}\varepsilon} e^{-s^2/2}\,ds, \tag{3.67}$$

making the same substitution as before. Hence, we have to take half of the previous result:

$$F(x) \sim \frac{1}{2}\sqrt{\frac{2\pi}{-x\phi''(c)}} f(a)e^{x\phi(a)} \quad \text{as } x \uparrow \infty. \tag{3.68}$$

Now suppose that $\phi''(c) = 0$. What happens now? We still need ϕ to have a maximum at c, so we had better assume that $\phi(t) \sim \phi(c) + \phi^{(p)}(c)(t-c)^p/p!$ as $t \to c$, with p even and $\phi^{(p)}(c) < 0$. Taking the same approximation of the interval of width 2ε around c gives

$$\begin{aligned}
F(x) &\sim \int_{c-\varepsilon}^{c+\varepsilon} f(c)\exp\left(x\left(\phi(c) + \frac{\phi^{(p)}(c)}{p!}(t-c)^p \right)\right) dt && (3.69)\\
&= \frac{(p!)^{1/p}}{(-x\phi^{(p)}(c))^{1/p}} f(c)e^{x\phi(c)} \int_{-(-x\phi^{(p)}(c)/p!)^{1/p}\varepsilon}^{(-x\phi^{(p)}(c)/p!)^{1/p}\varepsilon} e^{-s^p}\,ds && (3.70)\\
&\sim \frac{(p!)^{1/p}}{(-x\phi^{(p)}(c))^{1/p}} f(c)e^{x\phi(c)} \int_{-\infty}^{\infty} e^{-s^p}\,ds, && (3.71)
\end{aligned}$$

by making the substitution $s^p = -x\phi^{(p)}(c)(t-c)^p/p!$ as before. Crucially, the even power means that the remaining integral is double the one from 0 to ∞, so we can rewrite it as

$$F(x) \sim \frac{2(p!)^{1/p}}{(-x\phi^{(p)}(c))^{1/p}} f(c) e^{x\phi(c)} \int_0^\infty e^{-s^p}\, ds. \tag{3.72}$$

We can evaluate the integral exactly by substituting $u = s^p$, so $ds = u^{1/p-1}\, du/p$, and

$$\int_0^\infty e^{-s^p}\, ds = \frac{1}{p} \int_0^\infty u^{1/p-1} e^{-u}\, du = \frac{\Gamma(1/p)}{p} = \Gamma(1+1/p), \tag{3.73}$$

and so we conclude that[12]

$$F(x) \sim \frac{2\Gamma(1+1/p)(p!)^{1/p}}{(-x\phi^{(p)}(c))^{1/p}} f(c) e^{x\phi(c)}. \tag{3.74}$$

Finally, let us demonstrate the method for obtaining higher-order terms. We shall find the second term in the expansion; any further would be considerably more work. Suppose that we have a simple maximum at $c \in (a,b)$ as before. We have the same asymptotic approximations of f and ϕ, but extend them a couple more orders:

$$f(t) \sim f(c) + f'(c)(t-c) + \frac{1}{2} f''(c)(t-c)^2 \tag{3.75}$$

$$\phi(t) \sim \phi(c) + 0 + \frac{1}{2}\phi''(c)(t-c)^2 + \frac{1}{6}\phi'''(c)(t-c)^3 + \frac{1}{24}\phi''''(c)(t-c)^4; \tag{3.76}$$

we again assume $\phi''(c) < 0$. (We will need more terms than you might at first expect: remember that the integral is even about c.)

The idea is as follows: we can cope with the quadratic term in the exponential, so we shall retain that. The higher-order terms we expect to have lesser influence, so we make the approximation that $e^z \sim 1 + z + z^2/2$ for $c \to 0$; we will have a justification for this after carrying out the now-familiar substitution for t. So far, our hypotheses have led us to

$$F(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} \left(f(c) + f'(c)(t-c) + \frac{1}{2} f''(c)(t-c)^2 \right) \times \tag{3.77}$$

$$\exp\left(x \left(\phi(c) + \frac{1}{2}\phi''(c)(t-c)^2 + \frac{1}{6}\phi'''(c)(t-c)^3 + \frac{1}{24}\phi''''(c)(t-c)^4 \right) \right) dt, \tag{3.78}$$

with remainder exponentially suppressed. Now, set $s^2 = -x\phi''(c)(t-c)^2$ as usual. After collecting the xs, we find that the integral becomes

$$\frac{e^{x\phi(c)}}{(-x\phi''(t))^{1/2}} \int_{-(-x\phi''(c))^{1/2}\varepsilon}^{(-x\phi''(c))^{1/2}\varepsilon} \left(f(c) + \frac{f'(c)}{(-x\phi''(c))^{1/2}} s + \frac{f''(c)}{-2x\phi''(c)} s^2 \right) \tag{3.79}$$

$$\exp\left(\frac{1}{2} s^2 + \frac{\phi'''(c)}{6x^{1/2}(-\phi''(c))^{3/2})} s^3 + \frac{\phi''''(c)}{24x(-\phi''(c))^2} s^4 \right) dt \tag{3.80}$$

Now, supposing that ε is such that $x^{1/2}\varepsilon^3$ is small, we can expand the last two terms in the exponential in a power series, *viz.*

$$\exp\left(\frac{\phi'''(c)}{6x^{1/2}(-\phi''(c))^{3/2})} s^3 + \frac{\phi''''(c)}{24x(-\phi''(c))^2} s^4 \right) \tag{3.81}$$

$$\sim 1 + \left(\frac{\phi'''(c)}{6x^{1/2}(-\phi''(c))^{3/2})} s^3 + \frac{\phi''''(c)}{24x(-\phi''(c))^2} s^4 \right) + \frac{\phi'''(c)}{2 \cdot 6^2 x(-\phi''(c))^3)} s^6 \tag{3.82}$$

[12]Sadly we have to approximate $\Gamma(p/q)$ for $q > 2$, since there is no exact formula like there is for $q = 2$.

Now, the integral is over an interval symmetric about zero, so when any terms with odd powers of s will contribute nothing. Sticking this expansion into the integral, keeping terms that are $O(1/x)$, and extending the interval to the whole real line as before gives

$$\frac{e^{x\phi(c)}}{(-x\phi''(t))^{1/2}} \int_{-\infty}^{\infty} e^{-s^2/2}\left(f(c) + \frac{1}{x}\left(\frac{f''(c)}{-2\phi''(c)} + \frac{f'(c)\phi'''(c)}{6(-\phi''(c))^2}s^4\right.\right. \tag{3.83}$$

$$\left.\left. + \frac{f(c)\phi''''(c)}{24(-\phi''(c))^2}s^4 + \frac{f(c)(\phi'''(c))^2}{72(-\phi''(c))^3}s^6\right)\right)dt \tag{3.84}$$

Now we have to look at integrals of the form

$$\int_{-\infty}^{\infty} s^{2n}e^{-s^2/2}\,ds, \tag{3.85}$$

which we can either reduce to the Gamma-function, or integrate by parts a few times to evaluate as $\sqrt{2\pi}(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$. Sticking *these* into the expansion finally gives

$$F(x) \sim \sqrt{\frac{2\pi}{-x\phi''(t)}}e^{x\phi(c)}\left(f(c) + \frac{1}{x}\left(\frac{f''(c)}{-2\phi''(c)} + \frac{f'(c)\phi'''(c)}{2(-\phi''(c))^2} + \frac{f(c)\phi''''(c)}{8(-\phi''(c))^2} + \frac{f(c)(\phi'''(c))^2}{24(-\phi''(c))^3}\right)\right). \tag{3.86}$$

So yes, *considerably* more work.

Remark 19. It is possible to reformulate all of these integrals using the Inverse Function Theorem, reversion of series and Watson's lemma, but the calculation of the coefficients very quickly becomes so horrible that the extra effort is not worth it. So please don't.

3.3.2. Stirling's Formula

We are now equipped to give an easy proof of Stirling's approximation for the Γ-function, which we shall do in this section. Then, we shall struggle our way to the next term in the series as an example for the previous section.

Theorem 20 (Stirling's Formula). *The Γ-function satisfies the asymptotic*

$$\Gamma(n) \sim \sqrt{2\pi}n^{n-1/2}e^{-n}\left(1 + \frac{1}{12n}\right) \tag{3.87}$$

Proof. Obviously the idea is to apply Laplace's Method to the Γ-integral,

$$\Gamma(n) = \int_0^{\infty} t^{n-1}e^{-t}\,dt. \tag{3.88}$$

What is ϕ going to be? The large parameter is on t^{n-1}, so we'll have to reformulate the integral to put it in the right form. Set $t = nu$, so $dt/t = du/u$, and then write $u^n = e^{n\log u}$, so we have

$$\Gamma(n) = \int_0^{\infty} n^n \exp n(\log u - u)\,du. \tag{3.89}$$

Now we have an integral to which we can apply Laplace's Method: take $\phi(u) = \log u - u$, and we obtain

$$\phi'(u) = \frac{1}{u} - 1, \qquad \phi''(u) = -\frac{1}{u^2}, \tag{3.90}$$

and so the minimum is at $u = 1$, and $\phi(1) = -1$, $\phi''(1) = 1$, so the first-order approximation is

$$\Gamma(n) \sim \sqrt{\frac{2\pi}{n}} n^n e^{-n}, \tag{3.91}$$

as expected. Now, we have to go for the next-to-leading-order term. The easiest way to to this is to change variables again, then expand as a series in $1/n$. Let $u = 1 + s/\sqrt{n}$, and then

$$n\phi(1 + s/\sqrt{n}) = -n + 0 - \frac{s^2}{2} + \frac{s^3}{3\sqrt{n}} - \frac{s^4}{4n} + O(n^{-3/2}). \tag{3.92}$$

We now expand the exponential terms beyond the square to $O(1/n)$,

$$\exp\left(\frac{s^3}{3\sqrt{n}} - \frac{s^4}{4n} + O(n^{-3/2})\right) \sim 1 + \frac{s^3}{3\sqrt{n}} - \frac{s^4}{4n} + \frac{s^6}{2 \times 9n} + O(n^{-3/2}), \tag{3.93}$$

and integrate this after multiplying it by the lower-order part that we used before:

$$\Gamma(n) \sim n^{n-1/2} e^{-n} \int_{-\infty}^{\infty} e^{-s^2/2} \left(1 + \frac{s^3}{3\sqrt{n}} - \frac{s^4}{4n} + \frac{s^6}{2 \times 9n} + O(n^{-3/2})\right) ds. \tag{3.94}$$

The s^3 term is odd, so its integral vanishes, and the others may be done by parts, and we finally reach

$$\Gamma(n) \sim \sqrt{2\pi} n^{n-1/2} e^{-n} \left(1 + \frac{1}{12n}\right), \tag{3.95}$$

as required. □

Exercise Consider the integral

$$K_0(x) = \int_0^{\infty} \exp\left(-t - \frac{x^2}{4t}\right) \frac{dt}{t}, \tag{3.96}$$

which is a special solution to the modified Bessel equation of order 0,

$$x^2 y'' + xy' - x^2 y = 0, \tag{3.97}$$

which is of interest in some evanescent wave problems. Apply a similar analysis to obtain the first two terms of the large-x asymptotic expansion of this integral.

3.4. Oscillatory Integrals: The Method of Stationary Phase

We start with a significant result, with some conspicuous names attached to it:

Theorem 21 (Riemann–Lebesgue lemma). *Let $-\infty \leqslant a < b \leqslant \infty$, and let $f : [a, b] \to \mathbb{C}$ be an integrable function; that is, $\int_a^b |f(t)|\, dt < \infty$. Then*

$$\int_a^b e^{ixt} f(t)\, dt \to 0 \quad \text{as } |x| \to \infty. \tag{3.98}$$

You know a consequence of this theorem: if f is integrable, its Fourier coefficients decay to zero.

Exercise Prove using the Riemann–Lebesgue lemma that if f is k-times differentiable, its Fourier coefficients $\tilde{f}_n := \int_{-b}^{b} e^{\pi int/b} f(t)\,dt$ decay as $O(n^k)$.

Proof of the Riemann–Lebesgue lemma in the general case is both not required by us and not much to do with the other content of the course, so it is relegated to Appendix A.

We shall apply this theorem in order to approximate integrals of the form

$$\int_a^b f(t)e^{ix\phi(t)}\,dt, \quad \text{as } x \uparrow \infty \tag{3.99}$$

where x is real, and $f(t)$ and $\phi(t)$ are as usual continuous real-valued functions with at least some derivatives at every point. The simplest nontrival case is probably $\phi(t) = t$, in which case we can integrate by parts to find

$$F(x) = \int_a^b f(t)e^{ixt}\,dt = \left[f(t)\frac{e^{ixt}}{ix} \right]_a^b - \frac{1}{ix}\int_a^b f'(t)e^{ixt}\,dt. \tag{3.100}$$

Therefore if at least one of $f(a)$ and $f(b)$ is nonzero and $f'(t)$ is integrable, $F(x) = O(1/x)$ as $x \uparrow \infty$. However, what if the function $\phi(t)$ is more complicated? What can we say then? We begin with the simplest case.

Lemma 22. *Suppose that $\phi'(t)$ has one sign on $[a, b]$ (i.e. ϕ is monotonic on the interval). Then*

$$F(x) = \int_a^b f(t)e^{ix\phi(t)}\,dt \sim \frac{1}{ix}\left(\frac{f(b)}{\phi'(b)}e^{ix\phi(b)} - \frac{f(a)}{\phi'(a)}e^{ix\phi(a)} \right). \tag{3.101}$$

Proof. Without loss of generality, assume $\phi'(t) > 0$. Notice that ϕ is a differentiable bijection $[a, b] \to [\phi(a), \phi(b)]$ Therefore we may substitute $u = \phi(t)$, the inverse function rule gives $dt = \frac{du}{\phi'(\phi^{-1}(u))}$, and so

$$F(x) = \int_{\phi(a)}^{\phi(b)} \frac{f(\phi^{-1}(u))}{\phi'(\phi^{-1}(u))}e^{ixu}\,du. \tag{3.102}$$

Ah, but this is in the form we originally integrated by parts: doing so, we obtain the result. □

However, what happens if $\phi(t)$ has a stationary point in the interval? To understand this properly, we first consider

3.4.1. The Fresnel integrals

The integrals

$$C(x) := \int_{-\infty}^{\infty} \cos(xt^2)\,dt, \quad S(x) := \int_{-\infty}^{\infty} \sin(xt^2)\,dt, \tag{3.103}$$

are called Fresnel integrals. It is certainly not obvious that they even exist, since the integrands do not decay in any way as $|t| \to \infty$. However, the oscillations do become faster and faster as t increases, so we might expect that some form of alternating series argument might apply. A change of variables makes this more explicit: set $s = xt^2$, then $dt = ds/\sqrt{xs}$, and the integral $C(x)$ becomes

$$C(x) = \frac{2}{\sqrt{x}}\int_0^{\infty} \frac{\cos s}{\sqrt{s}}\,ds. \tag{3.104}$$

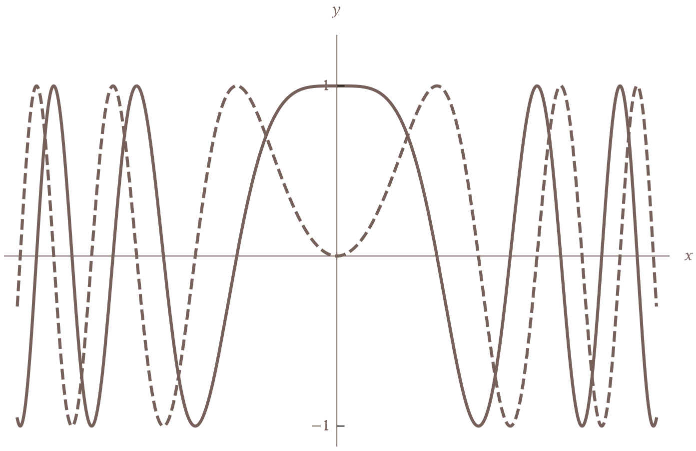

Figure 3.2.: Graphs of cos (t^2) (solid) and sin (t^2) (dashed)

Splitting the integral at $\pi/2, 3\pi/2, 5\pi/2, \ldots$, it is easy to show that the integrand is positive on the first, negative on the second, and alternates between positive and negative subsequently. Further, the integrand has absolute value bounded by $s^{-1/2}$, so the value of the integral over each of these intervals is decreasing. Hence the integral converges, by the alternating series test. A similar analysis can be carried out for $S(x)$, and in particular, we find that

$$C(x) = \frac{C(1)}{\sqrt{x}}, \quad S(x) = \frac{S(1)}{\sqrt{x}}. \tag{3.105}$$

There remains the question of evaluating $C(1)$ and $S(1)$. This can be done as follows: we know that

$$\frac{1}{\sqrt{s}} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s\lambda^2} \, d\lambda; \tag{3.106}$$

inserting this into the definition of $C(1)$, and interchanging the order of integration, we have

$$C(1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\int_0^{\infty} e^{-s\lambda^2} \cos s \, ds \right) d\lambda. \tag{3.107}$$

The inner integral is a straightforward integration by parts, evaluating to $1/(1+\lambda^4)$, so the full integral is reduced to

$$C(1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d\lambda}{1+\lambda^4}. \tag{3.108}$$

This integral is easy to do using partial fractions or calculus of residues, and we find that

$$C(1) = \sqrt{\frac{\pi}{2}}. \tag{3.109}$$

A similar calculation works for $S(1)$, and perhaps surprisingly, $S(1) = C(1)$. By adding and subtracting these integrals, we then have the result

$$\int_{-\infty}^{\infty} e^{\pm ixt^2} dt = C(x) \pm iS(x) = (1 \pm i)\sqrt{\frac{\pi}{2x}} = \sqrt{\frac{\pi}{x}} e^{\pm i\pi/4} = \sqrt{\frac{\pi}{\mp ix}}, \tag{3.110}$$

choosing the square root with $\sqrt{1} = 1$ and branch cut along the negative real axis. This tells us a couple of things:

1. These integrals do *not* decay as $1/x$.

2. A look at the graphs (Figure 3.2) shows the majority of the area is close to $t = 0$, where the function has an extended region (of width about $\sqrt{\pi/x}$) where it does not oscillate.[13]

Of course, we've considered these integrals because they are the archtype of oscillatory integrals where the 'phase' function $\phi(t)$ has a stationary point.

3.4.2. The Method of Stationary Phase

Therefore, let us now consider a more general phase function: let $\phi(t)$ have a single stationary point at $c \in (a, b)$. We have the usual

$$\phi(t) = \phi(c) + 0 + \frac{\phi''(c)}{2}(t-c)^2 + O((t-c)^3) \tag{3.111}$$

Let ε be a small number (that will depend on x). Then we have

$$\int_a^b f(t)e^{ix\phi(t)}\,dt = \int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{ix\phi(t)}\,dt + \int_{(a,b)\setminus(c-\varepsilon,c+\varepsilon)} f(t)e^{ix\phi(t)}\,dt. \tag{3.112}$$

Note that $\phi'(t) \neq 0$ away from c, so the second integral is, by our previous argument, $O(1/x)$. We should therefore focus on the first integral, since we expect it to contain a term with slower decay. In the small interval, we apply the Taylor approximation and substitute $s^2 = -x\phi''(c)/2(t-c)^2$ in a similar way to the last section:

$$\int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{ix\phi(t)}\,dt \sim \int_{c-\varepsilon}^{c+\varepsilon} f(c)e^{ix\phi(c)}e^{ix\phi''(c)(t-c)^2/2}\,dt \tag{3.113}$$

$$= e^{ix\phi(c)}\sqrt{\frac{2}{-x\phi''(c)}}\int_{-(-x\phi''(c)/2)^{1/2}\varepsilon}^{(-x\phi''(c)/2)^{1/2}\varepsilon} e^{-is^2}\,ds \tag{3.114}$$

$$\sim e^{ix\phi(c)}\sqrt{\frac{2}{-x\phi''(c)}}\int_{-\infty}^{\infty} e^{-is^2}\,ds = \sqrt{\frac{2\pi}{-ix\phi''(c)}}e^{ix\phi(c)}. \tag{3.115}$$

Example 7. Consider the function

$$J_0(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin t}\,dt, \quad r \geqslant 0 \tag{3.116}$$

which is a Bessel function of order 0.[14] Notice that the integral is over a whole period, so the endpoints are not important. There are two extrema, and we need to consider both. So,

$$\phi(t) = \sin t, \tag{3.117}$$
$$\phi'(t) = \cos t, \tag{3.118}$$
$$\phi''(t) = -\sin t, \tag{3.119}$$

so we have extrema at $t = \pi/2$ and $t = 3\pi/2$. For the first, the stationary phase formula gives

$$\sqrt{\frac{2\pi}{-rx\phi''(c)}}e^{ir\phi(c)} = \sqrt{\frac{2\pi}{ir}}e^{ir} = \sqrt{\frac{2\pi}{r}}e^{i(r-\pi/4}, \tag{3.120}$$

[13]Of course, this is a rather difficult thing to nail down, but the value of the integral speaks for itself.

[14]Which you met in Methods, when separating variables in polar coordinates.

(notice that in taking the square root, the correct convention is to take the principal value of the argument and halve it) while for the second, we get

$$\sqrt{\frac{2\pi}{-rx\phi''(c)}}e^{ir\phi(c)} = \sqrt{\frac{2\pi}{-ir}}e^{-ir} = \sqrt{\frac{2\pi}{r}}e^{-i(r-\pi/4)}, \tag{3.121}$$

and so the sum is simply

$$J_0(r) \sim \frac{2}{\sqrt{\pi r}}\cos\left(r - \frac{\pi}{4}\right). \tag{3.122}$$

There are several key differences between this and Laplace's method:

1. *All* of the stationary points contribute to the lowest-order term: the value of ϕ'' doesn't affect the order of the corresponding term in stationary phase.
2. Whereas in Laplace's method the endpoints of the interval contribute only exponentially small terms, the integration by parts that we did previously shows that here, the endpoints can be expected to contribute an $O(1/x)$. (And therefore we would have to do *even more* work to find the next term in the approximation)

So in general, we can expect the approximations we make using stationary phase to not be as accurate as for Laplace's method.[15]

3.5. Method of Steepest Descent

The above two methods deal with real functions, but you may be concerned about a couple of things:

1. The two Methods look rather similar, and use similar ideas. What connects them?
2. The Method of Stationary Phase is rather lame compared to Laplace's Method, especially regarding the error terms.
3. What about contour integrals? Can we approximate them in a similar way?

In this section, we aim to answer all of these. First, we have to discuss some ideas from complex analysis.

3.5.1. A detour into the complex plane

I shall not recap basic complex analysis here: you should all know what an analytic function is, and how a contour integral works (Cauchy's theorem and so on). What we are interested in is integrals of the form

$$F(x) = \int_C f(z)e^{x\phi(z)}\,dz, \tag{3.123}$$

where C is some curve in the complex plane, and $x \uparrow \infty$ is still real.

Recall the Cauchy–Riemann equations: if $\phi(p+iq) = u(p,q) + iv(p,q)$ is analytic, then

$$u_p = v_q, \quad u_q = -v_p. \tag{3.124}$$

[15]And this is again indicative of downward trend in the returns for our asymptotic approximations in this course.

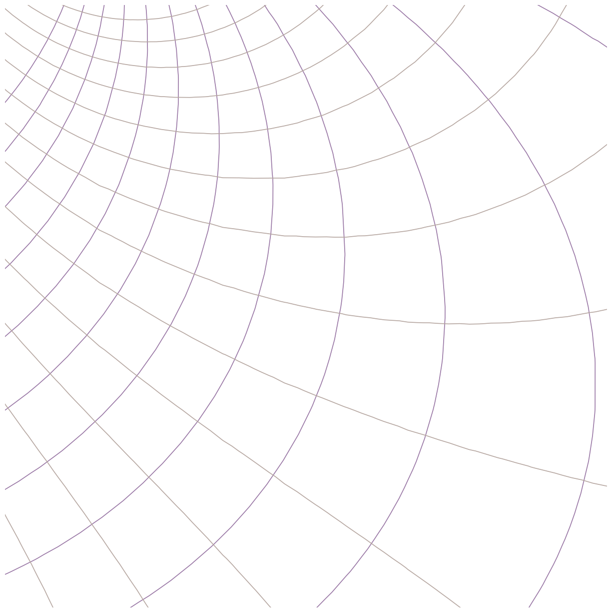

Figure 3.3.: The real and imaginary parts of a complex function have perpendicular contours, in the sense of level sets

Therefore, if we consider $\nabla u = (u_p, u_q)$ and $\nabla v = (v_p, v_q)$, the two-dimensional gradients of u and v, then

$$\nabla u \cdot \nabla v = u_p v_p + u_q v_q = -v_q u_q + u_q v_q = 0, \tag{3.125}$$

so the gradient vector fields are perpendicular everywhere. In particular, this means than any curve $v = \text{const.}$, which you will recall is perpendicular to ∇v, is parallel to the vector ∇u at any point.

Fine, but what's the point of all this? Look again at the integrand. Through experience, we know that the important part is the $e^{x\phi(z)}$. But the size of this term (i.e. the modulus) is determined by the real part of $\phi(z)$: we have

$$e^{x\phi(z)} = e^{xu(p,q)} e^{ixv(p,q)}, \tag{3.126}$$

and the latter term has absolute value 1. Further, Cauchy's theorem allows us to deform the contour of integration, providing we do not cross any points where f or ϕ are not analytic. Therefore, it is possible that the right choice of deformed contour will make the integral $F(x)$ easier to approximate well.

What should we look for to do this? Remember that the idea behind Laplace's method is that the integrand is exponentially smaller outside the intervals where the function in the exponential has a maximum. Therefore, we should look for contours along which the integrand becomes small as quickly as possible. Looking at the above reasoning, we notice that this means lines along which u decreases most rapidly. But we know what these are: curves parallel to ∇u, and moreover, these curves are precisely lines of constant v. These curves are often called *steepest descent contours for ϕ*.

There is another benefit in choosing such a contour: the integrand no longer oscillates increasingly rapidly in the way that makes the Fourier integrals so difficult to approximate.

There is one more problem: suppose that $\phi'(c) = 0$. Then there are multiple contours through c on which v is constant, but by the maximum principle for analytic functions, $\phi'(c)$ must be a saddle point. It follows that on some contours, u decreases as we move away from c, but on others, u increases.

Obviously to continue pursuing our idea, we will want to choose two on which u decreases.[16] If we must cross a ridge in u, this will also clearly also be the optimal point to do so, so we should try to pass through a stationary point of ϕ if available. An asymptotic approximation made in this way is a *saddle point approximation.*

We can now give a basic summary of the *Method of Steepest Descents*:[17]

1. Deform the integration contour to lie along curves of constant $\Im(\phi)$, preferably through saddle points c of ϕ such that the curve is a steepest descent contour away from c.

2. Parametrise the contour in the neighbourhood of the maxima of $\Re(\phi)$.

3. Use Laplace's method and/or Watson's Lemma to find the contributions from each endpoint and saddle point.

This is obviously not a very easy algorithm to execute: the arrangement of contours and saddles can be arbitrarily complicated, and the best way to get a feel for it is to do a few examples.

Example 8. We can improve on our stationary phase example by considering instead a steepest-descent contour, in the appropriate form: starting from (3.116), if we shift the interval of integration and apply symmetry, we find that

$$J_0(r) = \frac{1}{\pi}\int_0^{\pi} e^{ir\cos t}\,dt = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{iu}}{\sqrt{1-u^2}}\,du, \tag{3.127}$$

when we substitute $u = \cos t$. This still looks pretty awful, but if we plot the contours of iu, we find that the lines $\Re(u) = C$ are the steepest descent contours, and we should instead integrate around the other three sides of the rectangle with vertices at $1, 1+iT, -1+iT, -1$. (We choose the branch cut of $\sqrt{1-u^2}$ to lie in the lower half-plane, so that the function is real and positive on the real interval $(0,1)$.) By Cauchy's theorem, the integrals are equal, so

$$\pi J_0(r) = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}. \tag{3.128}$$

We start with γ_1, where we have $u = 1+it$, $0 < t < T$, so

$$\int_{\gamma_1} = \int_0^T \frac{e^{ir(1-it)}}{\sqrt{1-(1+it)^2}} i\,dt = ie^{ir}\int_0^T \frac{e^{-rt}}{\sqrt{t(t-2i)}}\,dt, \tag{3.129}$$

which we recognise as an integral we can apply Watson's Lemma to! Meanwhile, on γ_2, $u = t+iT$, $-1 < t < 1$, so

$$\int_{\gamma_2} = -\int_{-1}^{1} \frac{e^{-irt-rT}}{\sqrt{1-(-t+iT)^2}}\,dt, \tag{3.130}$$

but it is easy to see that this is bounded above by a multiple of e^{-rT}, and so tends to zero if we take $T \to \infty$. Lastly, on γ_3, we have $u = -1+it$ with $0 < t < T$, and

$$\int_{\gamma_3} = -ie^{-ir}\int_0^T \frac{e^{-rt}}{\sqrt{t(t+2i)}}\,dt, \tag{3.131}$$

[16] And perhaps it would be more appropriate to only call these curves 'steepest descent contours', but this is not necessarily universal.

[17] First considered for a specific function by Riemann, in an unpublished work *Sullo svolgimento del quoziente di due serie ipergeometriche in frazione continua infinita*, eventually published in his collected works [9], article XXIII, on hypergeometric continued fractions, and then done in general by Nekrasov, a Russian mathematician you've never heard of. See also the paper [7].

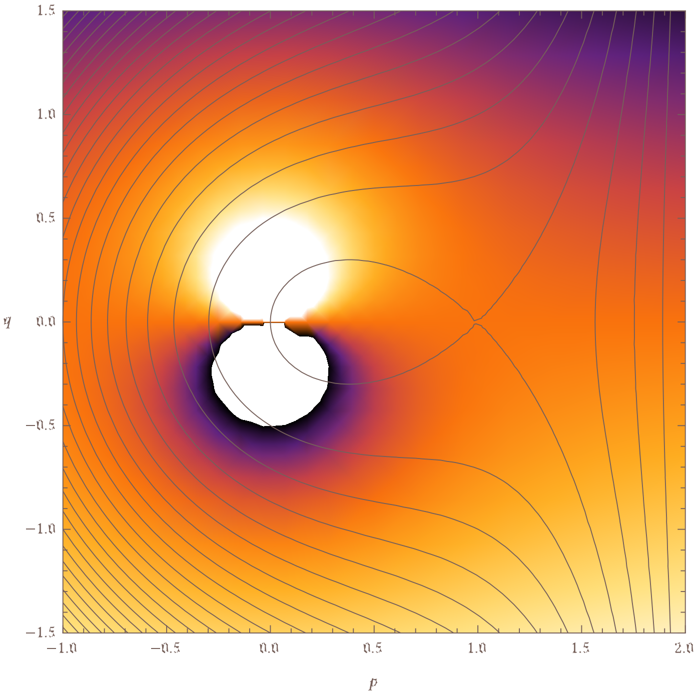

Figure 3.4.: Lines of constant $\Im(\phi)$ for $\phi(t) = i(t + 1/t)$. The shading shows the value of the real part, darker shades corresponding to more negative values.

which is similar to γ_1. Therefore, taking the limit in T, we have the sum of two Watson integrals,

$$\pi J_0(r) = ie^{ir} \int_0^\infty \frac{e^{-rt}}{\sqrt{t(t-2i)}}\, dt - ie^{-ir} \int_0^\infty \frac{e^{-rt}}{\sqrt{t(t+2i)}}\, dt \tag{3.132}$$

From here, we can derive the whole asymptotic expansion for $J_0(r)$ for $r \to \infty$.

Exercise Do this, and compare with the answer found in § 3.4.2.

Remark 23. The behaviour in the above example, although simple, is quite typical: if the endpoints of the integral are on different $\Im(\phi)$ contours, you have to find a way to join them with exponentially small error; this often means taking a contour to ∞ and back!

Example 9 (Saddle points). Consider

$$C(r) = \int_0^\infty \exp\left(ir\left(t + \frac{1}{t}\right)\right) dt. \tag{3.133}$$

This is nastier: if we calculate the imaginary part of $\phi(t) = i(t + 1/t)$, with $t = p + iq$, we find that the steepest descent contours are given by

$$\frac{p(1 + p^2 + q^2)}{p^2 + q^2} = c \tag{3.134}$$

(see Figure 3.4).

There are clearly two branches of the contour passing through 0, so we need to choose the right one. Then

$$\phi'(t) = i\left(1 - \frac{1}{t^2}\right), \tag{3.135}$$

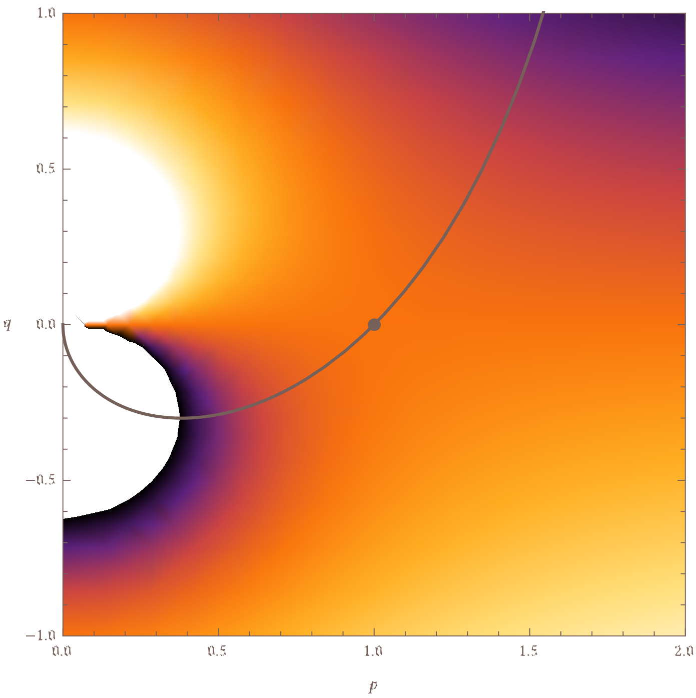

Figure 3.5.: The appropriate contour from Figure 3.4, passing through the saddle point $(1, 0)$ (marked) and into the small values of $\Re(\phi)$.

so there is a saddle point at $t = 1$. The contour that passes through $t = 1$ has $c = 1(1+1+0)/1 = 2$, and if we consider the real part, we find that the contour we should consider is the one that passes smoothly into the upper-right quarter-plane (Figure 3.5). The endpoints give exponentially small contributions, so we need to expand near the saddle point. There, we have locally $p = 1 + \alpha$, $q = 0 + \beta$, so

$$\beta^2 = -\frac{\alpha^2(1+\alpha)}{\alpha - 1} \sim \alpha^2, \tag{3.136}$$

so the curve that we are interested in has tangent $\beta = -\alpha$, which we can parametrise as $t = 1 + \frac{e^{i\pi/4}u}{\sqrt{r}}$. Then

$$\phi\left(1 + \frac{e^{i\pi/4}u}{\sqrt{r}}\right) = 2i + \frac{i^2}{r}u^2 + O(r^{-3/2}) \tag{3.137}$$

and the integrand becomes

$$\exp\left(r\left(2i - \frac{u^2}{r} + O(r^{-3/2})\right)\right) = e^{-2ir}e^{-u^2+O(r^{-1/2})}, \tag{3.138}$$

to which we can apply the usual Laplace argument to find

$$C(r) \sim \sqrt{\frac{\pi}{2r}}e^{2ir}e^{i\pi/4}, \tag{3.139}$$

where the last factor comes from changing variables in the integral.

The above examples give a sense of how to apply the method of steepest descents: in general, a saddle point at c contributes

$$\int_C e^{x\phi(t)}\, dt \sim \sqrt{\frac{2\pi}{-x\phi''(c)}}e^{x\phi(c)}, \tag{3.140}$$

by applying Laplace's Method at c.

Exercise Prove this, by considering a line through c tangent to a steepest descent contour.

In the next section, we shall apply steepest descents to a particular function, which we shall need in the rest of the course.

3.6. The Airy function

In the next section, we shall be interested in the solution of the differential equation

$$y'' = xy \tag{3.141}$$

and its asymptotics: such functions form a fairly universal way of studying differential equations with simple *turning points*). The equation is called the Airy equation, and its solutions are *Airy functions.*[18]

The equation is solved by

$$\int_C \exp\left(\frac{1}{3}t^3 + xt\right) dt, \tag{3.142}$$

where C is a contour that starts and ends at ∞, in the regions where the real part of t^3 diverges to $-\infty$.[19]

In particular, we can choose the contour so that one solution is

$$\mathrm{Ai}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(\frac{1}{3}it^3 + ixt\right)dt = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{1}{3}t^3 + xt\right)dt, \tag{3.143}$$

which is called *the* Airy function. This converges if x is real.

Exercise Show that

$$\mathrm{Ai}(0) = \frac{1}{3^{2/3}\Gamma(2/3)}, \quad \mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}. \tag{3.144}$$

Now, we need to bring this into the form $e^{x\phi(t)}$. At this point, it becomes apparent that the sign of x matters, since the substitution will involve taking a square root.

3.6.1. Asymptotics for $x > 0$

First we substitute $s = x^{1/2}t$. Then

$$\mathrm{Ai}(x) = x^{1/2}\int_{-\infty}^{\infty} \exp\left(ix^{3/2}\left(\frac{1}{3}s^3 + s\right)\right)dt, \tag{3.145}$$

and so

$$\phi(s) = \frac{1}{3}is^3 + is, \tag{3.146}$$

[18]Named after Airy, sometime Lucasian professor, astronomer, and thoroughly nasty chap.

[19]See Further Complex Methods for the derivation of this and the demonstration that two such solutions are linearly independent.

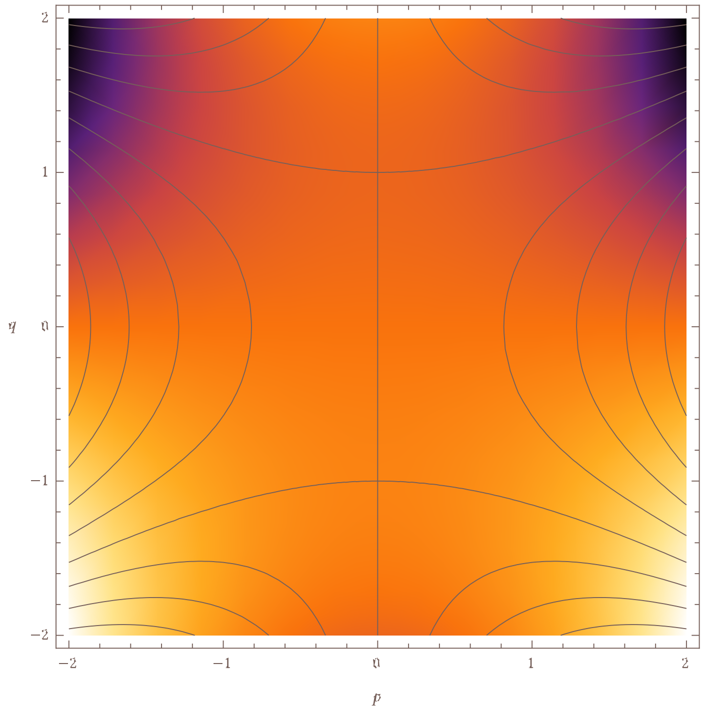

Figure 3.6.: Steepest descent contours for the Airy function

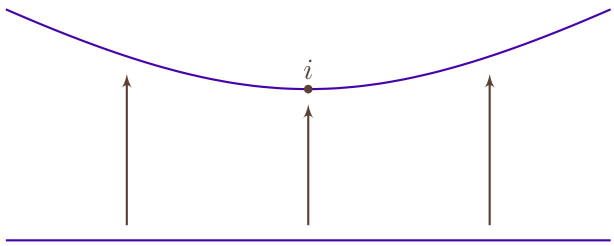

Figure 3.7.: Deformation of the Airy contour to the steepest-descent contour through the saddle point at $t = i$

and so

$$\phi(p+iq) = \left(\frac{1}{3}q^3 - p^2 q - q\right) + i\left(\frac{1}{3}p^3 - pq^2 + p\right). \tag{3.147}$$

Therefore the steepest descent contours are given by $\frac{1}{3}p^3 - pq^2 + p = C$. Further, $\phi'(s) = i(s^2+1)$, so the saddle points are at $s = \pm i$. It is easy to check that for $p = 0$, $q = 1$ we have $C = 0$. Therefore, we can plot the steepest descent contours, Figure 3.6.

Now, the method of steepest descents tells us to deform the contour to pass through i, and follow the steepest descent contour into the regions in the upper half-plane where $\frac{1}{3}q^3 - p^2 q - q$ decreases:

Now, we want to estimate the integral along this contour. Obviously the main contribution must be from the saddle point; the expansion of the function here is

$$\phi(i+z) = -\frac{2}{3} - z^2 + O(z^3). \tag{3.148}$$

The tangent $z = i + p$ at the saddle is a good approximation to the steepest descent contour, and hence we can apply our standard formula:

$$\mathrm{Ai}(x) \sim \frac{x^{1/2}}{2\pi} e^{x^{3/2}\phi(i)} \sqrt{\frac{2\pi}{-x^{3/2}\phi''(i)}} = \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-2x^{3/2}/3} \tag{3.149}$$

Exercise In fact, the simple nature of the phase function here allows us to obtain the full asymptotic expansion. Derive the next term. *And the rest of the expansion.

So the decay is superexponential. We will return to this later on.

3.6.2. Asymptotics for $x < 0$

If $x < 0$, we have to do a different substitution for x: set $s = (-x)^{1/2}t$, and then the exponent becomes

$$(-z)^{3/2} i \left(\frac{1}{3} s^3 - s \right). \tag{3.150}$$

This function has two saddle points, both on the real axis, at $s = \pm 1$.

We see that in fact we have here a stationary phase integral; since we only want the leading-order behaviour, it is sufficient to use the stationary-phase approximation. Then the expansions of $\phi(s) = s^3/3 - s$ at the stationary points are

$$\phi(1 + (s-1)) = -\frac{2}{3} + (s-1)^2 + O(s^3) \tag{3.151}$$

$$\phi(-1 + (s+1)) = \frac{2}{3} - (s+1)^2 + O(s^3) \tag{3.152}$$

which gives immediately

$$\mathrm{Ai}(-x) \approx \frac{(-x)^{1/4}}{2\pi} \sqrt{\frac{2\pi}{(-x)^{3/2}}} \left(e^{-2i(-x)^{3/2}/3} e^{i\pi/4} \sqrt{\frac{1}{2}} + e^{2i(-x)^{3/2}/3} e^{i\pi/4} \sqrt{\frac{1}{-2}} \right) \tag{3.153}$$

$$= \frac{1}{2\sqrt{\pi}} (-x)^{-1/4} (e^{2(-x)^{3/2}/3 - i\pi/4} + e^{2i(-x)^{3/2}/3 + i\pi/4}) \tag{3.154}$$

$$= \frac{1}{\sqrt{\pi}} (-x)^{-1/4} \cos\left(\tfrac{2}{3}(-x)^{3/2} - \tfrac{\pi}{4} \right) \tag{3.155}$$

as $x \to -\infty$.

Hang on, why have I written $\approx$ instead of $\sim$? Because this is not technically a true asymptotic expansion, because the cosine and the Airy function have zeros in slightly different places. Some more work would show that we can write

$$\mathrm{Ai}(-x) = \frac{1}{\sqrt{\pi}} (-x)^{-1/4} \cos\left(\tfrac{2}{3}(-x)^{3/2} - \tfrac{\pi}{4} \right) + O((-x)^{-7/4}), \tag{3.156}$$

but that's beyond the scope of this course to calculate in detail (but see the exercise below).

Oh, and before we move on, I'd just like to show you *how good* these asymptotic expansions are: see Figure 3.8.

Exercise Show using the saddle-point approximation that this agrees with the steepest-descent answer. Extend this to find the next term in the approximation.

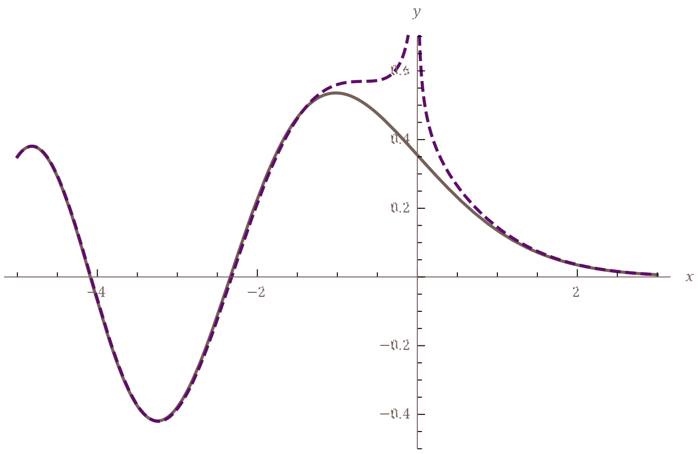

Figure 3.8.: Graph of the Airy function $\mathrm{Ai}(x)$ (solid) and its leading-order asymptotic approximations (dashed)

4. Approximating Solutions of Differential Equations

Here we must face up to two inevitable facts of life:

1. Sometimes we have to deal with bad old floppy differential equations, rather than nicely-nailed-down integrals.

2. Often, these differential equations do not have familiar solutions in terms of special functions.

We now move into the second part of the course, where we study differential equations and the approximation of their solutions. The fundamental idea shall be that some quantity is *slowly varying* compared to another one.[1]

What, in particular, are we interested in? Two (related) types of differential equation: those of the form

$$y''(x) = Q(x)y(x), \tag{4.1}$$

with Q very large, which may be approximately solved using the *Liouville–Green approximation*, and ones of the form

$$\varepsilon^2 y''(x) = q(x)y(x), \tag{4.2}$$

where ε is a small parameter, which are the realm of WKBJ.

Now, you may be wondering about the generality of such equations; if so, I recommend you tackle the following

Exercise Consider the general second-order linear differential equation

$$y'' + py' + qy = 0,$$

where p, q are functions. Show by substituting $y(x) = u(x)v(x)$ that this equation can be put in the form

$$u'' + Qu = 0,$$

where Q is a function of p and q.

4.1. Expansion about irregular singular points: the Liouville–Green approximation

Consider the differential equation

$$y'' = Qy. \tag{4.3}$$

[1]Examples include ray-tracing from Part II Waves, and classical reduction of the Schrödinger equation to the Hamilton–Jacobi equation.

Many years ago you learnt how to expand the solution to a differential equation about an *ordinary point*, where the coefficient Q is expandable in a power series, and, using the Method of Frobenius, *regular singular points*, where $x^2Q(x)$ is expandable in a power series. Hopefully, this has left you wondering what we do when Q is *more* singular, i.e. we have an *irregular singular point*.

To answer this question, we use what is called the Liouville–Green approximation: we try a solution of the form $y = e^S$, and derive an equation for S:

$$y'' = (S'' + S'^2)y, \tag{4.4}$$

so the equation is transformed to

$$S'' + S'^2 = Q, \tag{4.5}$$

which you may recognise as a Ricatti equation for S'. Now, the important assumption we make is that although Q is large, Q'/Q is small (we could say that Q is large, but varies slowly). Then if we try the first approximation

$$S_0' = \pm\sqrt{Q}, \tag{4.6}$$

the derivative of S_0', i.e. S_0'', will also be small compared to S_0':

$$S_0'' = \pm\frac{Q'}{\sqrt{Q}} \ll Q. \tag{4.7}$$

Now, as a second approximation, write $S = S_0 + S_1$, and we again hypothesise that S_1'' is much smaller than both S_0'' and $(S_0 + S_1)'^2$, and find that

$$(S_0 + S_1)' \sim \pm\sqrt{Q}(1 - S_0''/Q)^{1/2} = \pm Q^{1/2}\left(1 \mp \frac{Q'}{2Q^{1/2}}\right) \sim \pm Q^{1/2} - \frac{1}{4}\frac{Q'}{Q^{1/2}} \tag{4.8}$$

Now, we can easily solve this equation: it gives

$$S \sim \pm\int_a^x \sqrt{Q(x)}\,dx - \frac{1}{4}\log Q, \tag{4.9}$$

and so we find an approximation to y as the linear combination

$$y \approx AQ^{-1/4}\exp\left(\int\sqrt{Q}\right) + BQ^{-1/4}\exp\left(-\int\sqrt{Q}\right); \tag{4.10}$$

this is called the *Liouville–Green approximation* to the solution of (4.3).[2]

Remark 24. You've probably noticed that the above analysis looks rather sketchy compared to the techniques we are used to for integrals (cf. my warning at the beginning of the section), hence the $\sim$s have basically disappeared. This can all be laid on a more rigorous foundation, but unfortunately we don't have time to do so in this very short course. The places to look are Jeffreys's book *Asymptotic Approximations*, [4], Ch. 3, and Erdélyi's *Asymptotic Expansions*, [3], Ch. 4., where the remainder terms are studied using Volterra integral equations.[3] There is also a nice exposition in Olver's *Asymptotics and Special Functions*, [6], Chapter 6, § 2.

[2] We can check easily that the Wronskian of these solutions is indeed nonzero, so this can be an approximation of the general solution to (4.3).

[3] A Volterra integral equation is one of the form

$$x(t) = f(t) + \int_a^t K(t,s)x(s)\,ds,$$

where f, K are known functions; even though you haven't seen much of them in Tripos, there is a well-developed theory of how to solve them by repeated approximation (similar to the Dyson series used in Time-Dependent Perturbation Theory in Quantum Mechanics): in general, you should feel encouraged, rather than panicked, when you run into one of these.

Remark 25. It is possible to develop this idea to produce an asymptotic series solution; unfortunately the series is in general not convergent, so the irregular singular point case is genuinely nastier than the others.

Okay, let's look at an example or two.

Example 10 (Silly example). Let's begin with an example where we can calculate the solution exactly, to check that this method is a sensible one. The equation

$$y'' = \frac{a^2}{x^4}y, \quad a, x > 0 \tag{4.11}$$

has an irregular singular point at $x = 0$. We compute directly: suppose $y = e^S$, and $S'' \ll S'^2$, then

$$S' = \pm\frac{a}{x^2} \implies S = \mp\frac{a}{x} + \text{const.} \tag{4.12}$$

Substituting this back into the equation, taking $y = e^{S+C}$ with $C'' \ll C'^2 \ll S'^2$, we find

$$2S'C' = \frac{a^2}{x^4} - S'^2 - S'' \tag{4.13}$$

$$\pm\frac{2a}{x^2}C' = \frac{a^2}{x^4} - \frac{a^2}{x^4} - \pm\frac{-2a}{x^3} \tag{4.14}$$

$$C' = \frac{1}{x}, \tag{4.15}$$

and so $C = \log x + \text{const.}$ and we can write the solution as

$$y \approx Axe^{a/x} + Bxe^{-a/x}. \tag{4.16}$$

However, it is easy to check that the exact solution is given by

$$y = Axe^{a/x} + Bxe^{-a/x}, \tag{4.17}$$

which is exactly what we found with Liouville–Green.

Example 11 (Less silly example). Let's look at a similar example for which there is no elementary-function solution:

$$y'' = \frac{a^2}{x^3}y. \tag{4.18}$$

Setting $y = e^S$, we find

$$S'' + S'^2 = \frac{a^2}{x^3}. \tag{4.19}$$

Neglecting S'' gives

$$S' = \pm\frac{a}{x^{3/2}} \implies S = \mp\frac{2a}{x^{1/2}}, \tag{4.20}$$

and then the second approximation gives

$$2S'C' = S'' \tag{4.21}$$

$$\pm\frac{2a}{x^{3/2}}C' = \mp\frac{3a}{2x^{1/2}} \tag{4.22}$$

$$C' = -\frac{3}{4x} \implies C = -\frac{3}{4}\log x + \text{const.}, \tag{4.23}$$

and so the refined approximation to y is

$$y \approx Ax^{-3/4}e^{2ax^{-1/2}} + Bx^{-3/4}e^{-2ax^{-1/2}} \tag{4.24}$$

4.1.1. Irregular singular points at ∞

Many of the prevalent differential equations you have met, such as Bessel's equation and the Hermite equation, have irregular singularities at ∞ of the sort amenable to the Liouville–Green approximation. We can deal with these by changing variables, as you might expect: suppose that $Q(x)$ has an isolated irregular singular point at ∞. Then substituting $X = 1/x$, $Q(1/X)$ has an isolated irregular singular point at $t = 0$. Meanwhile, the derivatives transform as follows: set $Y(X) = Xy(1/X)$, so $y(x) = xY(1/x)$; then

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}\left(xY(1/x)\right) = \frac{d}{dx}\left(Y(1/x) - \frac{Y'(1/x)}{x}\right) = -\frac{Y'(1/x)}{x^2} + \frac{Y''(1/x)}{x^3} + \frac{Y'(1/x)}{x^2} \tag{4.25}$$

$$= \frac{Y''(1/x)}{x^3} = X^3 Y''(X), \tag{4.26}$$

so the differential equation becomes

$$Y'' = \frac{Q(1/X)}{X^3} Y, \tag{4.27}$$

which we can analyse as before.

Example 12 (Bessel's equation). Recall that Bessel's equation is given by

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right) y = 0; \tag{4.28}$$

we are interested in the large-x behaviour. Changing variables to eliminate the y' term: setting $y = uv$, we find that the coefficient of u' is

$$\frac{v}{x} + 2v', \tag{4.29}$$

which is zero if $v = x^{-1/2}$; the equation then reduces to

$$u'' = \left(-1 + \frac{n^2 - 1/4}{x^2}\right) u. \tag{4.30}$$

Now, we shall determine the nature of the singularity at $x = \infty$. Carrying out the $u = xU(1/x)$ substitution gives us

$$U'' = \left(-\frac{1}{X^4} + \frac{n^2 - 1/4}{X^2}\right) U, \tag{4.31}$$

and hence there is an irregular singular point at $X = 0$. To obtain the first-order approximation, we take the square root and find

$$\sqrt{Q} \sim \pm\frac{i}{X^2}, \quad Q^{-1/4} \sim e^{i\pi/4} X \tag{4.32}$$

$$\int \sqrt{Q}\, dX \sim \mp\frac{i}{X}, \tag{4.33}$$

and hence the lowest-order approximation is

$$U \approx AXe^{i/X} + BXe^{-i/X}. \tag{4.34}$$

Converting back to $y(x) = x^{1/2}u(x) = x^{-1/2}U(1/x)$ gives

$$y(x) \approx Ax^{-1/2}e^{ix} + Bx^{-1/2}e^{-ix}. \tag{4.35}$$

Remark 26. If you are happy to work with $Q(x)$ as $x \to \infty$, you may find it easier to not carry about the above substitution: you then find that $Q(x) \sim -1 + O(1/x^2)$, and so

$$u(x) \approx \sum_{\pm} Q^{-1/4} \exp\left(\pm \int \sqrt{Q(x)}\, dx\right) = Ae^{ix} + Be^{-ix}, \tag{4.36}$$

as in the Example's calculation.

Example 13 (The Airy equation). We've discussed one solution to the Airy equation. What are the asymptotics of a general solution to it? It is easy to show that

$$y'' = xy \tag{4.37}$$

has an irregular singular point at ∞. In this case, the functions are simple enough that we don't have to do the substitution we derived above. Let's look at $x > 0$: then the Liouville–Green solutions give

$$y \approx Ax^{-1/4} \exp\left(-\int \sqrt{x}\, dx\right) + Bx^{-1/4} \exp\left(\int \sqrt{x}\, dx\right) \tag{4.38}$$

$$= Ax^{-1/4} e^{-2x^{3/2}/3} + Bx^{-1/4} e^{2x^{3/2}/3}, \tag{4.39}$$

which tells us two things:

1. This agrees with our previous analysis of the Airy integral in the case that $A = 1/(2\sqrt{\pi})$, $B = 0$, so it seems at least plausible that our analysis gives the correct answer.

2. Solutions to the Airy equation in general blow up exponentially at $+\infty$: only one of them can decay at $+\infty$ (this is easily discovered by looking at the Wronskian, for example): hence our choice of Ai.

Meanwhile, for negative $x \to -\infty$, we have

$$y \approx Cx^{-1/4} e^{-2i(-x)^{3/2}/3} + Dx^{-1/4} e^{2i(-x)^{3/2}/3}, \tag{4.40}$$

which is the same sort of answer as we previously obtained.

Exercise Show that this analysis gives the same result as the method we derived at the beginning of the section.

A question you should ask yourself is: given the behaviour of a solution near $-\infty$, can I tell whether it decays at $+\infty$? Our analysis is not valid for the large region in the middle, so how can we obtain a formula that *connects* the different approximations of the same solution across regions where $Q(x)$ changes sign? We will investigate this shortly.

4.2. Equations with a large parameter: WKBJ

This is an application of the Liouville–Green approximation to derive approximate solutions and eigenvalues. It is especially useful in quantum mechanics.[4]

[4]Physicists call it WKB, after Wentzel, Kramers and Brillouin, the latter of whom you meet in the Applications of Quantum Mechanics course. The J is for Jeffreys, who has two claims to having his name on it: a) he developed it first, three years before W, K and B, and two years before the Schrödinger equation was even invented, and b) he's a Cambridge man.

We are interested in the equation

$$\varepsilon^2 y'' = qy, \tag{4.41}$$

where ε is very small.

In the previous section, we were interested in $y'' = Q(x)y$, when Q was large. It seems plausible that such approximations would be of the same form as those for this new equation: the second derivative y'' must be large compared to y, but now, the small parameter is ε, which is independent of x, so we might expect that the approximation will hold in extended regions.

Let's get on with it. We need to work out what form the asymptotic approximation will take. Start from our usual substitution, but this time introduce a small parameter δ, which we will use to see off the ε: set $y = e^{S(x)/\delta}$ (we assume that S and its first few derivatives are $O(1)$, i.e. normal-sized). Then (4.41) becomes

$$\varepsilon^2 \left(\frac{S''}{\delta} + \frac{S'^2}{\delta^2} \right) = q. \tag{4.42}$$

Somehow, we need to decide which term should dominate. We are expecting δ to be small, so there are really two possibilities: $\delta = \varepsilon$, and $\delta = \varepsilon^2$. It is easy to see that the latter is inconsistent: it gives you a S'^2/ε^2 term left over, which is very large. Try instead the former:

$$\delta = \varepsilon \qquad \Longrightarrow \qquad S'^2 + \varepsilon S'' = q. \tag{4.43}$$

Ah: now we're in business: the correction looks small in comparison to the other terms, and we have a Liouville–Green-type regime: we can take $S'^2 = q$, and have the first approximation

$$S \sim \pm \int_a^x \sqrt{q(t)}\, dt + O(\varepsilon) \tag{4.44}$$

$$y \approx \sum_{\pm} A_{\pm} \exp\left(\pm\frac{1}{\varepsilon} \int_a^x \sqrt{q(t)}\, dt + O(1) \right). \tag{4.45}$$

Here we obviously have a problem: $e^{O(1)}$ is not a small quantity! Therefore we really need to take a better approximation, that takes this into account.

Therefore, of course, we try $S = S_0 + \varepsilon S_1$, where S_0 is our previous solution with $S_0'^2 = q$:

$$q = \varepsilon^2 \left(\frac{(S_0' + \varepsilon S_1')^2}{\varepsilon^2} + \frac{S_0'' + S_1''}{\varepsilon} \right) = S_0'^2 + \varepsilon(S_0'' + 2S_0'S_1') + O(\varepsilon^2), \tag{4.46}$$

and equating the coefficients of ε gives

$$S_0'^2 = q, \qquad\qquad 2S_0'S_1' + S_0'' = 0; \tag{4.47}$$

the latter we know all about, the former gives

$$S_1' = \frac{S_0''}{2S_0'} = \tfrac{1}{2}(\log S_0')', \tag{4.48}$$

and therefore we find that the solution is

$$y \approx \sum_{\pm} A_{\pm} \exp\left(\pm\frac{1}{\varepsilon} \int_a^x \sqrt{q(t)}\, dt - \frac{1}{2}\log\sqrt{q} + O(\varepsilon) \right) \tag{4.49}$$

$$= \sum_{\pm} A_{\pm} q^{-1/4} \exp\left(\pm\frac{1}{\varepsilon} \int_a^x \sqrt{q(t)}\, dt \right)(1 + O(\varepsilon)), \tag{4.50}$$

which is the proper first-order approximation.

Remark 27. We could go on to higher order: because the error term in (4.43) looks $O(\varepsilon)$, we suspect that a possible form of solution is

$$y(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right), \tag{4.51}$$

where δ is some power of ε. If we suppose that $\delta = \varepsilon$ and substituting in, we have the equation

$$\left(\sum_{n=0}^{\infty} \varepsilon^n S_n'\right)^2 + \varepsilon \sum_{n=0}^{\infty} \varepsilon^n S_n'' = q; \tag{4.52}$$

we can then equate terms order-by-order in ε to obtain a consistent expansion. It is easy to show (see Example Sheet 3) that

$$\begin{aligned} S_0'^2 &= q \\ 2S_0'S_1' + S_0'' &= 0 \\ 2S_0'S_n' + \sum_{k=0}^{n-1} S_k'S_{n-j}' + S_{n-1}'' &= 0 \quad (n > 1), \end{aligned} \tag{4.53}$$

so our guess does indeed produce a consistent solution.

4.2.1. Turning points

At a point a where $q(a) = 0$, the approximations we have found don't work any more: one way to see this is simply that $q^{-1/4}$ blows up at a.

(An alternative way to see it is to recall that we need $|S_0'|^2 \gg \varepsilon\, |S_1'S_0'| \ll$, or

$$|q|^{3/2} \gg \varepsilon\, |q'|\,, \tag{4.54}$$

and we can't have this control if $q \to 0$.)

Therefore, if we want to model a solution near a turning point, we need an alternative strategy. Suppose that $q(a) = 0$. If $q'(a) = \mu \neq 0$, we call a a simple turning point. In this case, local to a the differential equation looks like

$$\varepsilon^2 y''(x) = \mu(x-a)y(x), \tag{4.55}$$

or if we substitute $z = (\frac{\mu}{\varepsilon^2})^{1/3}(x-a)$,

$$\frac{d^2y}{dz^2} = zy \tag{4.56}$$

But this is just the Airy equation![5] We have an *exact* solution of this equation, of which we know the asymptotics.

Now, for the vast majority of applications of this idea, we want to match to an exponentially decaying solution, so we will only look at that case. In particular, suppose we have our q, continuous, with a single turning point, at a. Suppose also that $\mu > 0$ (if not, we can just replace x with $-x$). Then for $x > a$ we have $q > 0$, and the approximation

$$y_+ \approx Aq^{-1/4} \exp\left(-\frac{1}{\varepsilon}\int_a^x \sqrt{q(t)}\, dt\right), \tag{4.57}$$

[5] And this is why we spent all that time on the Airy function in the last chapter. Surprise!

and for $x < 0$, $q < 0$ and we have the approximation

$$y_- \approx C(-q)^{1/4} \cos\left(\frac{1}{\varepsilon}\int_x^a \sqrt{-q(t)}\,dt - \gamma\right), \tag{4.58}$$

which we have primarily chosen because we are looking for a real-valued solution: we have written it as a cosine anticipating the asymptotics of the Airy function.

Okay, now we need to match it onto the Airy function. Taking x close to a, we use the approximation $q(t) \sim \mu(x-a)$ to derive:

$$y_+ \approx \frac{A}{(\mu(x-a))^{1/4}} \exp\left(-\frac{\sqrt{\mu}}{\varepsilon}\int_a^x (t-a)^{1/2}\,dt\right) = \frac{A}{(\mu(x-a))^{1/4}} \exp\left(-\frac{\sqrt{\mu}}{\varepsilon}\frac{2}{3}(x-a)^{3/2}\right) \tag{4.59}$$

$$y_- \approx \frac{A}{(\mu(a-x))^{1/4}} \cos\left(\frac{\sqrt{\mu}}{\varepsilon}\int_x^a (a-t)^{1/2}\,dt - \gamma\right) = \frac{A}{(\mu(a-x))^{1/4}} \cos\left(\frac{\sqrt{\mu}}{\varepsilon}\frac{2}{3}(a-x)^{3/2} - \gamma\right) \tag{4.60}$$

Now, the large-z asymptotics of the decaying solution to (4.56) are

$$y_0 \approx \frac{1}{2\sqrt{\pi} z^{1/4}} \exp\left(-\tfrac{2}{3} z^{3/2}\right) \tag{4.61}$$

$$= \frac{1}{2\sqrt{\pi}((\mu/\varepsilon^2)^{1/3}(x-a))^{1/4}} \exp\left(-\frac{2}{3}\frac{\sqrt{\mu}}{\varepsilon}(x-a)^{3/2}\right) \quad (x > a) \tag{4.62}$$

$$y_0 \approx \frac{1}{\sqrt{\pi}(-z)^{1/4}} \cos\left(\tfrac{2}{3}(-z)^{3/2} - \tfrac{1}{4}\pi\right) \tag{4.63}$$

$$= \frac{1}{\sqrt{\pi}((\mu/\varepsilon^2)^{1/3}(a-x))^{1/4}} \cos\left(\frac{2}{3}\frac{\sqrt{\mu}}{\varepsilon}(a-x)^{3/2} - \tfrac{1}{4}\pi\right) \quad (x < a) \tag{4.64}$$

Now, y_+ can easily be matched onto y by choosing A suitably; nothing difficult about that. Clearly, the way to match y to y_- is to take $\gamma = \frac{1}{4}\pi$. Remarkably, we also see that $A = C$, so the final matching of y_+ to y_- does not depend on the solution near the turning point at all! We have derived the *connection formulae*:

$$y_+(x) = C(q(x))^{-1/4} \exp\left(-\frac{1}{\varepsilon}\int_a^x \sqrt{q(t)}\,dt\right) \qquad x > a, x - a \gg \varepsilon^{2/3}; \tag{4.65}$$

$$y_0(x) = 2\sqrt{\pi}(\mu\varepsilon)^{-1/6} C \operatorname{Ai}(\varepsilon^{-2/3}\mu^{1/3}(x-a)) \qquad |x-a| \ll 1 \tag{4.66}$$

$$y_-(x) = 2C(-q(x))^{-1/4} \cos\left(\frac{1}{\varepsilon}\int_x^a \sqrt{-q(t)}\,dt - \tfrac{1}{4}\pi\right) \qquad x < a, a - x \gg \varepsilon^{2/3} \tag{4.67}$$

(The coefficient C remains undetemined, since we only imposed one condition on the solution: that it decay at $+\infty$.)[6]

Remark 28 (More general matching). We take Bi to be a solution of Airy's equation that is asymptotically orthogonal to Ai: viz,

$$\operatorname{Bi}(x) \sim -\frac{1}{(-x)^{1/4}\sqrt{\pi}} \sin\left(\tfrac{2}{3}(-x)^{3/2} - \tfrac{1}{4}\pi\right), \quad \text{as } (x \to -\infty). \tag{4.68}$$

[6]For a method of producing a uniform approximation which does not have the singularity at the turning point, see Appendix C.

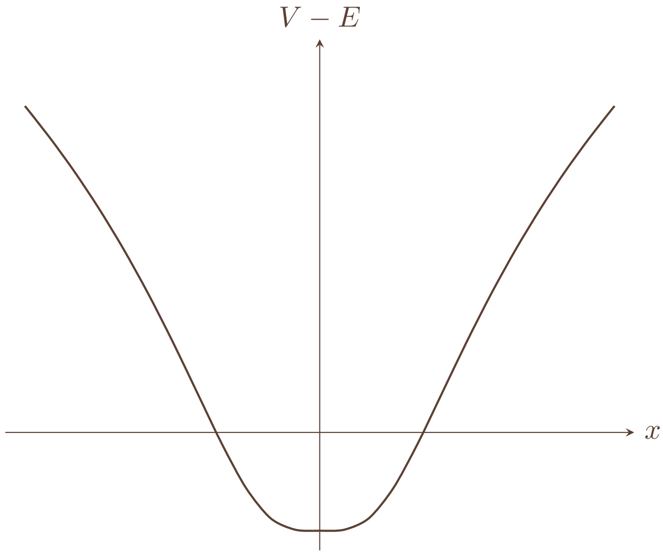

Figure 4.1.: A continuous potential that takes every positive value twice

Then we also have the asymptotic for positive x,

$$\mathrm{Bi}(x) \sim \frac{1}{x^{1/4}\sqrt{\pi}} \exp\left(\tfrac{2}{3}x^{3/2}\right) \tag{4.69}$$

With this, we can find a more general matching formula across a turning point:

$$\begin{aligned} y_+(x) &= C(q(x))^{-1/4} \exp\left(-\frac{1}{\varepsilon}\int_a^x \sqrt{q(t)}\,dt\right) && (4.70)\\ &\quad + D(q(x))^{-1/4} \exp\left(\frac{1}{\varepsilon}\int_a^x \sqrt{q(t)}\,dt\right) && x > a, x - a \gg \varepsilon^{2/3};\\ y_0(x) &= 2\sqrt{\pi}(\mu\varepsilon)^{-1/6} C\,\mathrm{Ai}(\varepsilon^{-2/3}\mu^{1/3}(x-a))\\ &\quad + \sqrt{\pi}(\mu\varepsilon)^{-1/6} D\,\mathrm{Bi}(\varepsilon^{-2/3}\mu^{1/3}(x-a)) && |x-a| \ll 1 \qquad (4.71)\\ y_-(x) &= 2C(-q(x))^{-1/4} \cos\left(\frac{1}{\varepsilon}\int_x^a \sqrt{-q(t)}\,dt - \tfrac{1}{4}\pi\right)\\ &\quad - D(-q(x))^{-1/4} \sin\left(\frac{1}{\varepsilon}\int_x^a \sqrt{-q(t)}\,dt - \tfrac{1}{4}\pi\right) && x < a, a - x \gg \varepsilon^{2/3} \qquad (4.72) \end{aligned}$$

This is exactly what Rayleigh, Gans, and finally Jeffreys did.[7]

4.2.2. Two turning points and bound states

Suppose now that we have a sort of well potential, which has two turning points, $a < b$, where $q(x) > 0$ for $x \notin [a, b]$, and there is a finite constant $c > 0$ so that $q(x) > c$ for all sufficiently large $|x|$. In other words, we're looking at the sort of situation in Figure 4.1

We would like to obtain a *bound state* solution of this system: the solution should be square-integrable, which means that we require the solution to decay at infinity. There are then three regions to consider:

[7]All before W, K or B stuck their heads above the parapet.

Region 1 $x < a$. The solution decays at least exponentially; we expect it to be described by

$$y_1 \sim Aq^{-1/4} \exp\left(-\frac{1}{\varepsilon}\int_x^a \sqrt{q(t)}\,dt\right) \tag{4.73}$$

Region 3 $b < x$. The solution is again exponentially decaying, described by

$$y_3 \sim Bq^{-1/4} \exp\left(-\frac{1}{\varepsilon}\int_b^x \sqrt{q(t)}\,dt\right) \tag{4.74}$$

Region 2 $a < x < b$. We have two ways of writing the solution, from matching onto Region 1 or Region 3. *Then*, we have to find a way for these approximations to match up. We have

$$y_{21} \approx 2A(-q)^{-1/4} \cos\left(\frac{1}{\varepsilon}\int_a^x \sqrt{-q(t)}\,dt - \tfrac{1}{4}\pi\right) \tag{4.75}$$

$$y_{23} \approx 2B(-q)^{-1/4} \cos\left(\frac{1}{\varepsilon}\int_x^b \sqrt{-q(t)}\,dt - \tfrac{1}{4}\pi\right) \tag{4.76}$$

These have to be equal for all x; certainly we need $A = \pm B$, and then if we write y_{23} as

$$2B(-q)^{1/4} \cos\left(\left(\frac{1}{\varepsilon}\int_a^b \sqrt{-q(t)}\,dt - \tfrac{1}{2}\pi\right) - \left(\frac{1}{\varepsilon}\int_a^x \sqrt{-q(t)}\,dt - \tfrac{1}{4}\pi\right)\right) =: 2B(-q)^{1/4}\cos(\phi - \theta), \tag{4.77}$$

where ϕ and θ label the brackets, we find that we have possible equations of the form $\cos\theta \pm \cos(\phi-\theta) = 0$. This can be rewritten to encompass both cases as

$$2\cos^2(\phi-\theta) - 2\cos^2\theta = \cos 2(\phi-\theta) - \cos 2\theta = 2\sin(2\theta-\phi)\sin\phi; \tag{4.78}$$

therefore, since ϕ depends on x, the only way to have this equation satisfied everywhere in the interval is to satisfy the condition

$$\frac{1}{\varepsilon}\int_a^b \sqrt{-q(t)}\,dt = \left(n + \tfrac{1}{2}\right)\pi, \tag{4.79}$$

where $n \in \{0, 1, 2, \dots\}$ since the integral should be positive. (We then find we also have $A = (-1)^n B$). This, then, is the condition required for $\varepsilon^2 y'' = q(x)y$ to have a bound state, to leading order in ε.[8]

Well, that's odd. That looks like a quantisation condition, doesn't it? Could there perhaps be a reason for that?

4.2.3. Application to Quantum Mechanics: estimate of energy levels

Now, one obvious example of a second-order equation with a small parameter on the second derivative is, of course, the (time-independent) Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi. \tag{4.80}$$

If we assume that Planck's constant is small relative to everything else (i.e. that the energy, the potential or the mass is much larger, as in a large system or a smaller system with high energy), then

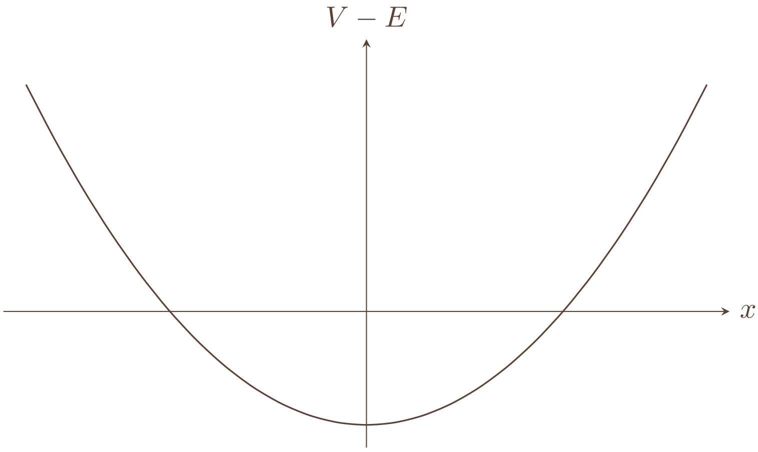

Figure 4.2.: A well-shaped potential

we can apply the JWKB approximation. We assume that V takes on some sort of generalised "well" shape, so for any E in a specified range, $V - E$ has two roots (Figure 4.2).

Therefore, identify $\varepsilon = \hbar$ and $q = 2m(V - E)$, and suppose that the two roots of q are $a < b$ (note that they in general depend on E). For a bound state, which must decay exponentially outside $[a, b]$, we must by the previous section have

$$\int_a^b \sqrt{2m(E - V(x))}\, dx = \left(n + \tfrac{1}{2}\right) \pi\hbar, \tag{4.81}$$

for $n \in \{0, 1, 2, \dots\}$. This is called the *Bohr–Sommerfeld* quantisation criterion. Therefore we obtain a discrete set of possible energies, $E_0, E_1, E_2, \dots$. The JWKB approximation is especially good for large n, since for large n the q term becomes larger in comparison to $\hbar$.

Okay, so what does this actually mean? Let's do some physics on this.

1. The interval $[a, b]$ is where the particle would live classically. The evanescent waves on the other parts of the axis are typical of how quantum mechanics behaves in regions adjoining the classically allowed region: *cf.* a particle trapped in a finite square well.

2. Recall that the classical energy is given by $E = \frac{p^2}{2m} + V$. In classical mechanics, the particle "orbits" by oscillating between a and b. Since we have $p^2 = 2m(E - V)$, we deduce that

$$\oint p\, dx = (2n + 1)\pi\hbar, \tag{4.82}$$

where the integral is over one orbit. Cf. the discussion of this integral in Classical Dynamics.

3. The local wavelength of the particle is the increment of the argument of the cosine, i.e. $(-q)^{1/2}/\varepsilon = 2\pi/\lambda$ (Cf. plane waves e^{ikx} for where this comes from: k is the frequency). Then we find

$$\lambda = \frac{2\pi\hbar}{p}, \tag{4.83}$$

which is as usual the de Broglie wavelength.

[8]Improvements to this can be made using an estimate due to Dunham: see Bender and Orszag, [1], § 10.7, p. 537f.

4. On the other hand, the *amplitude* of the squared-wavefunction locally, after averaging, is given by

$$|\psi(x)|^2 \propto ((-q)^{1/4})^2 = \frac{1}{p}, \tag{4.84}$$

i.e. proportional to the amount of time the particle would spend in the vicinity if it were orbiting classically.

These properties are essentially why this method is also called the *semiclassical approximation*: most of the interpretation ties into the classical probability density on phase space. These, interestingly, are the methods of Old Quantum Theory, which was surplanted by the 'new' Quantum Mechanics of Schrödinger and Heisenberg in the 1920s.[9]

Harmonic and generalised oscillators

Of course, if we're talking about physics, we can't avoid discussing the harmonic oscillator. We recall that this has (classical) Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 \tag{4.85}$$

on the phase space (x, p), and so the orbits in phase space with definite energy E are on the ellipse

$$\frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 = E; \tag{4.86}$$

we know that the area of this ellipse is $\pi\sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}} = 2\pi E/\omega$. Hence the Bohr–Sommerfeld energy levels are

$$\frac{2\pi E_n}{\omega} = 2\pi\left(n + \tfrac{1}{2}\right)\hbar \tag{4.87}$$

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega, \quad n \in \{0, 1, 2, \dots\}. \tag{4.88}$$

Surprisingly, these agree exactly with the true quantised energy levels![10] Also, the spacing between the energy levels here is $\hbar\omega$.

Suppose now we look at a more general oscillator, where V is a function that takes every positive value twice, and diverges as $|x| \to \infty$, like Figure 4.1. Then the phase portrait of the orbits of constant E will look like Figure 4.3.

We don't necessarily have a nice expression for the area contained by an orbit any more, but what we can do is look at the derivative with respect to E:

$$\frac{d}{dE}\oint p\,dx \approx \oint \left.\frac{\partial p}{\partial E}\right|_x dx = \oint \frac{m}{p}\,dx = \oint \frac{dt}{dx}\,dx = \oint dt = T(E), \tag{4.89}$$

the period associated with E, which we also can't find exactly unless the system is soluble.

Now, a word about the $\approx$: we have neglected the change in the orbit path in phase space that comes from changing E. Since this section is just a sketch, I'm just going to direct you to the calculation in David Tong's Classical Dynamics lecture notes [10], p. 110.

[9] Although we shall not do so, it is possible to show that applying this approximation procedure the energy levels of Hydrogen, for example, produces results close to the full nonrelativistic quantum mechanical predictions.

[10] It is possible to show that all higher-order corrections really do vanish identically for the harmonic oscillator, primarily due to the potential being a polynomial of low degree.

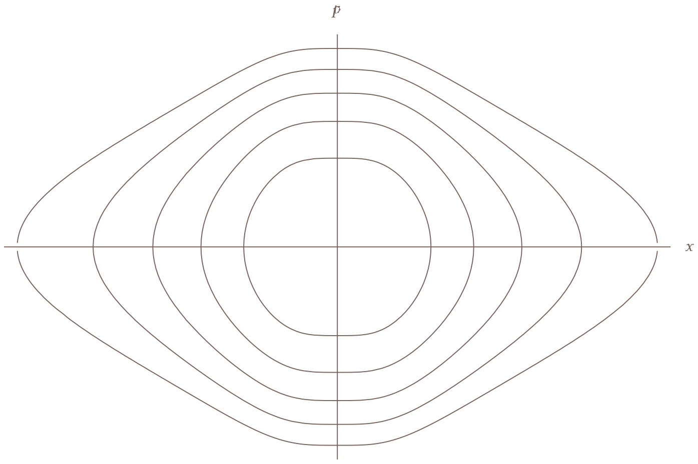

Figure 4.3.: Phase portrait of a general oscillator

Pretending that $\hbar$ is small, the energy level spacing is small,

$$\Delta \oint p\, dx \approx T(E)\Delta E. \tag{4.90}$$

But of course, we know what the left-hand side is: it's $2\pi\hbar$, by our quantisation condition. Hence

$$\Delta E \approx \frac{2\pi\hbar}{T(E)} =: \omega(E)\hbar, \tag{4.91}$$

where $\omega(E)$ is then the average angular frequency.

What's the point of this? If we have a charged particle in the $(n+1)$th state which drops to the nth state, basic quantum theory tells us it spits out a photon of quantised energy $\tilde{\omega}\hbar$ and frequency ω. But our calculation and conservation of energy imply that $\tilde{\omega} = \omega(E_n)$, which makes the classical partical theory consistent with the quantised frequencies observed in, say, the photoelectric effect.

Right, now we've had an obligatory plunge into physics, let's wash ourselves thoroughly and return to mathematics.

5. Recent Developments

5.1. Stokes returns

First, let us do a more thorough examination of the topic we were discussing in section 2.4, now that we have a lot more (and more complicated) functions to play with.

Suppose we have an asymptotic approximation for a function f in the form

$$f(z) \sim g(z) \quad \text{as } |z| \uparrow \infty. \tag{5.1}$$

This is in general only valid for a limited range of $\arg z$. Why? Because we have the (exact) relationship

$$f = g + (f - g), \tag{5.2}$$

and so the asymptotic condition requires us to have $f - g = o(f)$.

$f - g$ is called *recessive*, and g the *dominant* approximation to f. At a *sector boundary*, $f - g$ becomes the same order as g, and on the other side, $f - g$ dominates. These boundaries we call *Stokes lines*.

Now that we have the *Liouville–Green* approximations, we can see many applications of this: given an approximation of the form

$$f(z) \sim Ae^{S_1(z)} + Be^{S_2(z)}, \quad \text{as } |z| \to \infty, \tag{5.3}$$

such as Liouville–Green would give us, suppose that in some sector, $\Re S_2 \ll \Re S_1$. Then e^{S_2} is recessive, and often not even visible in the asymptotic series (being exponentially suppressed).

Now define the *Stokes lines* to be where $\Re S_2 = \Re S_1$.

Now, in general, f is supposed to be an entire function, but given how S_1, S_2 are defined, they are often *not* entire, due to the square roots. Hence even if we keep both A and B as not zero, we cannot expect the approximation (5.3) to be true globally. We can find a way around this, by taking A and B to be different in different sectors.

The best way to do this is to change the coefficient when it's making least difference to the approximation: i.e., when the difference between the real parts of S_1 and S_2 is largest. Often (but not always), this occurs when $\Im(S_1(z) - S_2(z)) = 0$; such curves are called *anti-Stokes lines.*[1]

Example 14 (Airy function, again). An archtypal example of this is, of course, the Airy function.[2] Recall the Airy equation,

$$y'' = zy, \tag{5.4}$$

and our two solutions, Ai and Bi. The first approximation to the solutions, using L–G, are $e^{2z^{3/2}/3}$ and $e^{-2z^{3/2}/3}$. Therefore the Stokes lines are where $\Re(z^{3/2}) = 0$, i.e.

$$\arg z \in \{\tfrac{1}{3}\pi, \pi, \tfrac{5}{3}\pi\}; \tag{5.5}$$

whereas the anti-Stokes lines, where $\Im(z^{3/2}) = 0$, are

$$\arg z = 0, \tfrac{2}{3}\pi, \tfrac{4}{3}\pi. \tag{5.6}$$

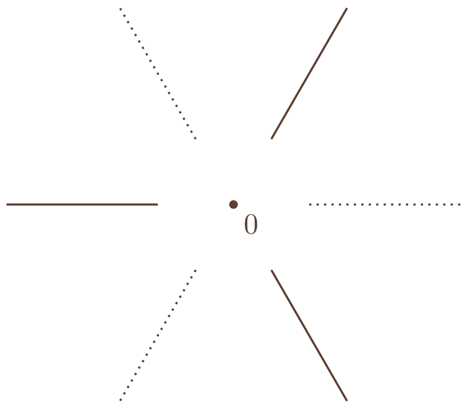

Figure 5.1.: Stokes lines (solid) and anti-Stokes lines (dotted) for solutions to the Airy equation (it is conventional to draw them not reaching the origin, since they apply to the large-$|z|$ asymptotics)

See Figure 5.1.

Now, let's go a bit further in the approximation. It is easy to show[3] that a general solution to Airy's equation is given by

$$y(z) \sim Az^{-1/4}e^{-2z^{3/2}/3}\left(1 - \frac{5}{48}z^{-3/2} + \cdots\right) + Bz^{-1/4}e^{2z^{3/2}/3}\left(1 + \frac{5}{48}z^{-3/2} + \cdots\right). \tag{5.7}$$

We want to find the A and B so that this is a good approximation to the Airy function Ai. To start with, we have the obvious choice

$$A = \frac{1}{2\sqrt{\pi}}, \quad B = 0, \quad |\arg z| < \pi : \tag{i}$$

this does what we know Ai does on the positive real axis.[4] The asymptotic result is correct, but the approximation itself is actually rather poor in the region $2\pi/3 < \arg z < \pi$.

To match on the negative real axis, we need a trick: clearly Ai, Bi are two linearly independent solutions to the the Airy equation. But the equation has the symmetry $z \mapsto \omega z$, where ω is a nontrivial cube root of unity (it doesn't matter which, in fact). Therefore $\operatorname{Ai}(\omega z)$ and $\operatorname{Ai}(\omega^2 z)$ are also solutions, and so must be expressible in terms of Ai and Bi. It turns out that

$$\operatorname{Ai}(z) = -\omega \operatorname{Ai}(\omega z) - \omega^2 \operatorname{Ai}(\omega^2 z), \tag{5.8}$$

which is easy to verify by using the series expansions at zero. Using this and our first asymptotic relation allows us to derive a second approximation and its associated region:

$$A = \frac{1}{2\sqrt{\pi}}, \quad B = \frac{i}{2\sqrt{\pi}}, \quad \tfrac{1}{3}\pi < \arg z < \tfrac{5}{3}\pi, \tag{ii}$$

[1] As I mentioned before, there is some dispute between mathematicians and physicists over which is which between Stokes and anti-Stokes lines. I promise I will be consistent, but make no such guarantees for the rest of the literature.

[2] I did tell you it was the course's pet function …

[3] And you are asked to do so on Example Sheet 3.

[4] Notice that in fact this approximation extends beyond the Stokes lines at $\pm\frac{1}{3}\pi$: this is unusual. See Bender and Orszag, [1], p. 133ff. for a justification.

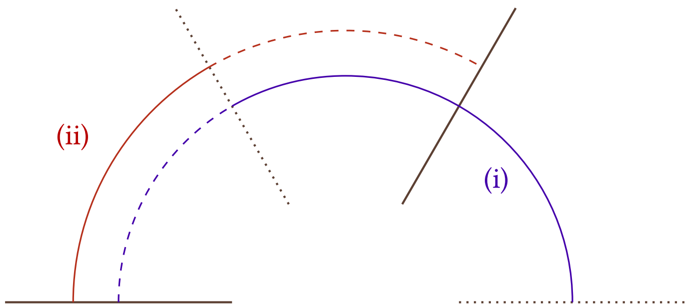

Figure 5.2.: Regions of validity for the approximations (i) and (ii) of $\mathrm{Ai}(z)$ in the upper half-plane. Solid indicates good approximation for finite z, dashed indicates asymptotic correctness (but poor approximation for finite z).

which you can check agrees with the formula for Ai that we found for the negative real axis, when the branch cuts of the $z^{3/2}$ are chosen correctly. Finally, using the reflection principle, $\mathrm{Ai}(z) = \overline{\mathrm{Ai}(\bar{z})}$, so we have the approximation in the final region:

$$A = \frac{1}{2\sqrt{\pi}}, \quad B = -\frac{i}{2\sqrt{\pi}}, \quad -\tfrac{5}{3}\pi < \arg z < -\tfrac{1}{3}\pi. \tag{5.9}$$

We can summarise these results with Figure 5.2.

5.1.1. Accuracy of approximation

Suppose we have a finite, fairly small $|z|$. How good are these approximations? Let's look at a simple example: $\mathrm{Ai}(z)$, with $\frac{2}{3}|z|^{3/2} = 1.5$. In Figure 5.3 are shown the exact curve, the leading-order asymptotic from the first region, the leading-order asymptotic from the second region, and finally the optimally truncated series, with a changeover on the anti-Stokes line $\arg z = 2\pi/3$.[5]

5.2. *Optimal truncation and Beyond

Right, with the examinable content from the course complete, let's have some fun[6]

5.2.1. Optimal truncation: what's goin' off?

We have mentioned before that the "best" approximation from an asymptotic series normally occurs by truncating the series at the smallest term. Why does this work? Let's look at a venerable example: the Stieltjes integral, which we found was equal to

$$\int_0^\infty \frac{e^{-t}}{1+tx}\,dt = \sum_{n=0}^{N} (-1)^n n! x^n + (-1)^N (N+1)! x^{N+1} \int_0^\infty \frac{e^{-t}}{(1+xt)^{N+2}}\,dt, \tag{5.10}$$

and since the integral is bounded by 1, we concluded that the sum gives an asymptotic series. Now, what is the smallest term in the sum? Fortuitously, none are zero, so we can divide them:

$$1 < \frac{(n+1)! x^{n+1}}{n! x^n} = (n+1)x, \tag{5.11}$$

[5]For those of you reading in black-and-white, the pink is next to the green.
[6]Your notion of fun may differ from that of the lecturer.

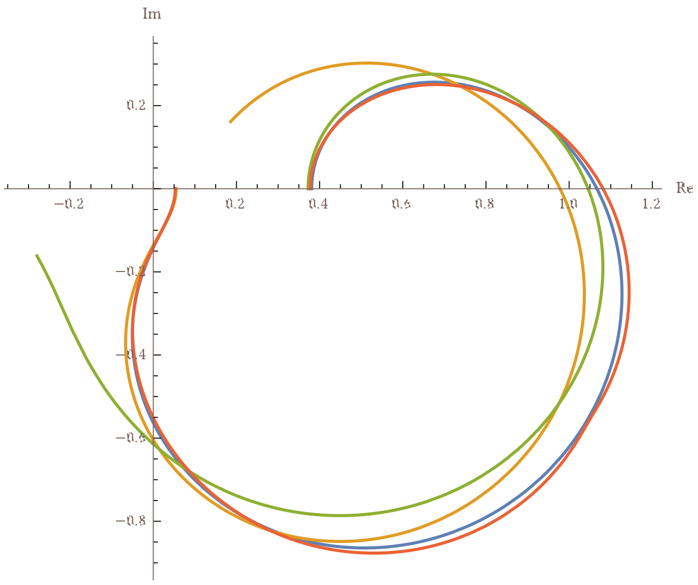

Figure 5.3.: Plots of $\mathrm{Ai}(z)$ and approximations on the semicircle $\frac{2}{3}\left|z\right|^{3/2} = 1.5$, $0 < \arg z < \pi$. In particular, includes: (■) the exact function, (■) the leading-order asymptotic (i) in its sector of asymptotic validity, (■) the leading-order asymptotic (ii) in its sector of asymptotic validity, and (■) the optimal truncation of the full series, with changeover on the anti-Stokes line.

so the smallest term is when $n = \lfloor x \rfloor$. We can then use Stirling's approximation to show that the error is

$$\left|R_{\lfloor 1/x \rfloor}(x)\right| \sim \sqrt{\frac{\pi}{2x}} e^{-1/x}, \quad \text{as } x \downarrow 0, \tag{5.12}$$

so optimal truncation produces an exponentially small error term! This is why optimal truncation is regarded as so useful.[7]

This is essentially the endpoint of classical asymptotics: in what Michael Berry called *superasymptotics*; it is also known as 'asymptotics beyond all orders'.

5.2.2. Resurgence and Hyperasymptotics

> *Les séries divergentes sont en général quelque chose de bien fatal et c'est une honte qu'on ose y fonder aucune démonstration.*[8]
>
> Niels Henrick Abel, 1826

Whenever anyone talks about anything related to divergent series, they always quote Abel as their straw man. I say, give the poor kid a break: he wasn't around long enough to see what could be done with them.[9]

[7] Unfortunately, actually proving that the optimal truncation error actually behaves like this is very difficult for most asymptotic series.

[8] Often somewhat loosely rendered as "Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever."

[9] And go and look up Abelian functions: they're supremely cool, and supremely important in nineteenth century mathematics. You've never heard of them because Riemann basically sorted out the entire subject.

This section will be more sketchy than the previous ones: the intention is to give an outline of the process, rather than a mathematically sound deduction.[10]

The story so far: the theory of asymptotic expansions that we have studied in this course is initially due to Poincaré: the main idea has been to truncate the asymptotic series at fixed N, and ask about behaviour as $x \to \infty$. Stokes, he of the lines, had had a different idea some 60 years earlier: for a given finite x, what is the best approximation we can get out of the series by truncating it? This gives us the *optimal truncation* after $N(z)$, which, as we discussed in the previous section, tends to have exponential accuracy, far outstripping Poincaré's expectations of power-law accuracy. This is also called *superasymptotics*, and for a long time, this was though to be the best one could wring from the asymptotic series.[11]

But, in the 1970s and '80s, physicists and mathematicians including Robert Dingle and others decided that

1. Exponential convergence is just not good enough, and
2. Surely the tail of the asymptotic series is good for something?

so they investigated ways to improve convergence by 'resumming' the tail of the series.

We start in the late nineteenth century. Darboux showed that a lot of functions of the type we were considering earlier, the high derivatives diverge like the factorial. As we know, the high derivatives pop up in the higher-order terms in basically any local expansion of a function: the simplest example we have studied of this is of course Watson's lemma. This suggests a certain "universality" in the behaviour of power series, and hence their asymptotic approximations.[12]

Meanwhile, we also have the following useful trick: Borel noticed that sometimes you can cheat such a divergent series by replacing the factorials by the integral form of the Gamma-function: *viz.*, we have the equalities

$$\sum_{r=0}^{\infty} \frac{a_r}{z^r} = \sum_{r=0}^{\infty} \frac{a_r}{r!} \frac{r!}{z^r} = \int_0^{\infty} e^{-t} \left(\sum_{r=0}^{\infty} \frac{a_r}{r!} \left(\frac{t}{z} \right)^r \right) dt, \tag{5.13}$$

at least for finite sums and uniformly convergent ones. It turns out that the integral on the right may actually have a finite value, even if the left-hand side is formally infinite: we can see this, for example, in the case $a_r = (-1)^r r!$, for which the right-hand side becomes

$$\int_0^{\infty} e^{-t} \left(\sum_{r=0}^{\infty} \left(-\frac{t}{z} \right)^r \right) dt = \int_0^{\infty} \frac{e^{-t}}{1 + t/z} \, dt, \tag{5.14}$$

which is up to a change of parameter our old friend the Stieltjes integral, which we know converges. This trick is called *Borel summation*; it turns out that it is the correct summation method to use here, because it has no problem with passing over Stokes lines.

[10]We also owe some structure to the article on divergent series in the new *Princeton Companion to Applied Mathematics*, pp. 634–640.

[11]There is also the question of obtaining "the function with a given an asymptotic expansion": $f(x)$ has a unique asymptotic expansion, but is there a sensible, well-defined way of producing a unique function with a given asymptotic expansion? Your immediate answer should be "What about adding an e^{-kx} term?", which as we know, is far smaller than any negative power of x, so will not appear in the usual asymptotic series, and spoils naïve uniqueness. Thankfully there is a way around this by enforcing particular conditions on the function that we want, which the extra negative exponential does not satisfy, but we'll say no more about this.

[12]As a slightly silly example, how many analytic functions do you know of with a finite radius of convergence? This is precisely the same phenomenon.

So we have a pile of series with terms that diverge factorially, and a way of assigning sensible values to divergent series. It should be fairly obvious where we're going now.

The tail of the divergent asymptotic series that we chopped off to obtain optimal truncation often has factorially-divergent terms. Therefore, we expect that we can use Borel summation to give it a sensible finite value in terms of an integral.

This is where Robert Dingle comes in. He made a number of key observations: firstly, that since the asymptotic series is obtained from the integral or differential equation in a well-defined way, it must somehow contain "all the information" about the solution functions, which we should expect to be able to disentangle from it. Second, he applied the idea of Borel-summing the tail after optimal truncation to hang on to the rest of the series. And third, and most crucially, he realised that in cases where a function has several different asymptotic series that might represent it (the familiar case being the Airy function's two exponentials), since each series completely characterises the function, there it must contain information about the other asymptotic series in its higher terms. This notion eventually came to be called *resurgence*.

Furthermore, he found a way to actually exemplify this relation using the natural variable F, the difference between the exponents (in the Airy case, this is $F = 4z^{3/2}/3$): using this, he was able to obtain characteristic and universal expansions for the terms of one series from the terms of the other. These can then be used to *resum* the tail of the series using the Borel approach, which reduces the general case to evaluating a small number of *terminant* integrals, which apply to many functions. Even better, the process can be repeated, and the error becomes exponentially smaller at each stage. This is now called *hyperasymptotics*.

It turns out, surprisingly, that iterating this summation and expansion of the error, the number of terms needing reworking approximately halves every time, so the iteration actually terminates! The famous diagram in Figure 5.4 shows the output of this process for a particular case of the Airy function.

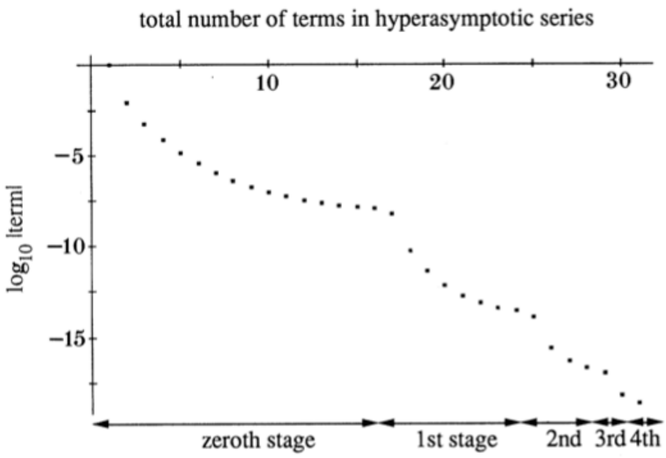

Figure 5.4.: Hyperasymptotics for the Airy function, at $F = 16$. Each extra sum contains about half as many terms as the preceeding one, and the final one only includes one extra term. The final error is approximately 15 orders of magnitude smaller than that from optimal truncation![13]

Another interesting side-effect of the terminant integrals is that there is a universal formula for how the approximation changes as one passes over a Stokes line (this is naturally incorporated into the theory, since the Stokes line corresponds to $\Im(F) = 0$): rather than appearing suddenly as we cross the Stokes line, it really does "[...] enter [...] into a mist, [...] and comes out with its coefficient changed."

[13]Image from M.V. Berry and C.J. Howls, *Hyperasymptotics*, Proc. R. Soc. Lond. A (1990) **430**, 653–667.

as Stokes himself put it many years later. In particular, the coefficient of the subdominant term changes like

$$\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{\Im(F)}{\sqrt{2\Re(F)}}\right)\right), \tag{5.15}$$

so the Stokes line has an effective, nonzero "width" of $\sqrt{2\Re(F)}$; the jump in the coefficient is only absolute at the origin, which is not in the domain we can consider with this idea.[14]

There is still *a lot* of current work on this, in particular in the more complicated examples of multiple integrals, and even functional integrals in quantum field theory. This seemingly very nineteenth-century, gears-and-wheels-and-grease mechanical theory is still a hot research topic![15]

5.3. Closing remarks

You may have noticed that the results in this course, although powerful, are quite simple; they also only apply to certain functions, specifically ones that can be massaged into exponential forms.

Your mathematical intuition should therefore tell you that this course is only the beginning of a long story, which has become much harder and more complicated as time moves on. To see some more modern techniques, which can handle a wider range of integrals and differential equations, the spiritual successor to this course is the Part III course *Perturbation and Stability Methods*,[16] which introduces Padé approximants, method of multiple scales, matched asymptotic expansions, …, all of which generalise concepts that we have discussed in this course.

For gritty physics that can be handled with these sorts of techniques, you have several options: Statistical Physics uses leading-order asymptotics in its thermal limits (passing from a quantum-mechanical theory to a statistical one being a large-N approximation, for example), and in Part III, perturbative expansions of integrals play vital rôles in the Quantum Field Theory courses—and it is known that these expansions are inevitably asymptotic, rather than convergent. Understanding the relationship between the sizes of the terms in the asymptotic expansion and the physical output of the system is one of the most important parts of 'physical' QFT;[17] this is where the dirty words *regularisation* and *renormalisation* come in.

Now, you pure mathematicians dozing at the back, I also have good news for you. Between them, Hardy, Ramanujan and Rademacher were able to produce an *exact, convergent* asymptotic expansion for the partition function, which involves bounding and calculating integrals in much the same way that we have done in this course, although the calculations involved are considerably more complex.

I hope you have found this course interesting and useful: I certainly have, and I hope it has shown you that approximating things is a bit more sophisticated than just lopping terms off the end of a Taylor series.

[14] This is actually not without controversy: some mathematicians dispute that this is a valid approximation to make, questioning the neglected error.

[15] And Martin Kruskal was convinced that the divergent series could be proven to come from a unique function by using John Conway's famously as-yet-unused surreal numbers. Sadly Kruskal died in 2006 before his many years' work on this subject produced any conclusive results. Other people have taken up the fight, but progress appears limited.

[16] Title and content may vary: it may just be called Perturbation Methods, but should cover similar generalisations.

[17] As opposed to what interests mathematicians in QFT: the hard maths you neet to formulate it sensibly.

A. *The Riemann–Lebesgue Lemma

We shall here fulfil our promise of a general proof of the Riemann–Lebesgue lemma. This divides into several key ideas:

1. approximation of integrable functions by simpler functions,
2. proof of the Lemma for simpler functions (trivial, set as an exercise in the main text),
3. bounding the error in the approximation.

The first step here is actually the most interesting: it depends somewhat on our definition of integral (it will turn out that this actually makes no difference in the long run, but depending on the definition, the proof is slightly different).

We'll give separate proofs for the Riemann/Darboux integral and the Lebesgue integral.[1]

Proof for Darboux integral. We have to take a, b finite for the Riemann integral. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. We define the upper and lower approximations to f as the step functions that give the upper and lower sums for P:

$$M(t) = \begin{cases} \sup_{x_i < t \leqslant x_{i+1}} f(t) & x_i < t \leqslant x_{i+1} \end{cases} \tag{A.1}$$

$$m(t) = \begin{cases} \inf_{x_i < t \leqslant x_{i+1}} f(t) & x_i < t \leqslant x_{i+1} \end{cases} \tag{A.2}$$

It is then clear by definition that

$$\int_a^b m(t)\, dt = U(f, P) \leqslant \int_a^b f(t)\, dt \leqslant L(f, P) = \int_a^b M(t)\, dt. \tag{A.3}$$

Because f is Riemann-integrable, for any $\varepsilon > 0$ we can find a partition P such that $0 \leqslant U(f, P) - L(f, P)\frac{1}{2}\varepsilon$. Hence

$$\left| \int_a^b e^{itx}(f(t) - m(t))\, dt \right| \leqslant \int_a^b |f(t) - m(t)|\, dt \leqslant \int_a^b (M(t) - m(t))\, dt = U(f, P) - L(f, P) < \tfrac{1}{2}\varepsilon. \tag{A.4}$$

Now, for this P, since $m(t)$ is a step function we can choose an X so that

$$\left| \int_a^b e^{itx} m(t)\, dt \right| < \tfrac{1}{2}\varepsilon. \tag{A.5}$$

Therefore it follows by using the triangle inequality that for $|x| > X$,

$$\left| \int_a^b e^{itx} f(t)\, dt \right| \leqslant \left| \int_a^b e^{itx}(f(t) - m(t))\, dt \right| + \left| \int_a^b e^{itx} m(t)\, dt \right| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon, \tag{A.6}$$

so $\int_a^b e^{itx} f(t)\, dt \to 0$ as $x \to \infty$, as required. □

[1]Mainly to advertise how much better at this the Lebesgue integral is.

In the Lebesgue case, we have the good fortune of having *defined* the integral so that our integrable functions are closely approximated by step functions, so we can just get on with the main proof:

Proof for the Lebesgue integral. This is *much* easier: we can prove it for each of the four steps of defining $L^1(\mathbb{R})$.

1. If f is the characteristic function of a finite interval, we compute directly

$$\int_c^d e^{itx}\,dt = \frac{1}{ix}(e^{idx} - e^{icx}), \tag{A.7}$$

   which obviously tends to 0 since the right-hand side is bounded above by $2/x$.

2. If f is a linear combination of characteristic functions of intervals (i.e. a step function), then linearity of the integral implies that the result extends.

3. If f is a positive integrable function, by definition there is an increasing sequence f_n of step functions converging to f almost everywhere, and $\int |f - f_n| = \int (f - f_n) \to 0$. Therefore we can choose N so that

$$\int (f - f_N) < \tfrac{1}{2}\varepsilon. \tag{A.8}$$

   Moreover, since $f_n(t)e^{itx} \to f(t)e^{itx}$ almost everywhere and $\left|f_n(t)e^{itx}\right| < |f(t)|$, the Dominated Convergence Theorem implies that $f(t)e^{itx}$ is integrable.

   The result for step functions implies that we can choose X so that

$$\left|\int f_N(t)e^{itx}\,dt\right| < \tfrac{1}{2}\varepsilon \tag{A.9}$$

   for every $|x| > X$. Finally, we just use the triangle inequality:

$$\begin{aligned} \left|\int f(t)e^{itx}\,dt\right| &\leqslant \left|\int (f(t) - f_N(t))e^{itx}\,dt\right| + \left|\int f_N(t)e^{itx}\,dt\right| && \text{(A.10)} \\ &\leqslant \int |f - f_N| + \tfrac{1}{2}\varepsilon \leqslant \varepsilon, && \text{(A.11)} \end{aligned}$$

   which proves the proposition for positive functions.

4. Finally, we extend to $L^1(\mathbb{R})$ in the usual way, by writing f as a sum of positive and negative, real and imaginary parts. $\square$

B. *Other Asymptotic Methods

In this appendix we shall briefly discuss some asymptotic methods that we do not have time to investigate in lectures.

B.1. The Euler–Maclaurin formula

One thing we have not touched on at all is the asymptotics of sums. Why?

One reason is that sums do not occur nearly so frequently in physics as their continuous friends.

A slightly more practical reason is that we have the Euler–Maclaurin formula:

Theorem 29 (Euler–Maclaurin formula). *Let $f : [a, b] \to \mathbb{R}$ be a $2N$-times differentiable function. Then*

$$\sum_{n=a}^{b} f(n) - \int_a^b f(x)\,dx = \frac{1}{2}\left(f(b) + f(a)\right) + \sum_{m=1}^{N} \frac{B_{2m}}{(2m)!}\left(f^{(2m-1)}(b) - f^{(2m-1}(a)\right) + R_N, \tag{B.1}$$

where R_N is given by

$$\frac{1}{(2N)!}\int_0^1 \phi_{2N}(t)\left(\sum_{m=0}^{b-a-1} F^{(2n)}(a+m+t)\right) \tag{B.2}$$

Therefore knowing how the integral behaves tells you an awful lot about how the sum behaves.

To prove this, we use as quick a method as possible:[1] take ϕ to be a polynomial of degree n, then for $0 < t < 1$, we have the following result due to Darboux:

$$\begin{aligned}\frac{d}{dt}&\sum_{m=1}^{n} (-1)^m (z-a)^m \phi^{(n-m)}(t) f^{(m)}(a + t(z-a)) \\ &= -(z-a)\phi^{(n)}(t) f'(a + t(z-a)) + (-1)^n (z-a)^{n+1}\phi(t) f^{(n+1)}(a + t(z-a))\end{aligned} \tag{B.3}$$

Since ϕ is a polynomial of degree n, $\phi^{(n)}(t) = \phi^{(n)}(0)$ is constant. If we integrate from 0 to 1 in t, we obtain

$$\begin{aligned}\phi^{(n)}(0)\left(f(z) - f(a)\right) = &\sum_{m=1}^{n} (-1)^{m-1}\left(\phi^{(n-m)}(1) f^{(m)}(z) - \phi^{(n-m)}(0) f^{(m)}(a)\right) \\ &+ (-1)^n (z-a)^{n+1} \int_0^1 \phi(t) f^{(n+1)}(a + t(z-a))\,dt\end{aligned} \tag{B.4}$$

If we take $\phi(t) = (t-1)^n$ in this, we obtain Taylor's theorem, but we have a new use for this identity. First we need a

[1] See, for example, Whittaker and Watson, [13], p. 125ff., although note their different convention for the Bernoulli numbers.

Definition 30 (Bernoulli polynomials). Recall that the *Bernoulli numbers* are given by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=1}^{\infty} B_n \frac{t^n}{n!}. \tag{B.5}$$

Similarly, the *Bernoulli polynomials* are given by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=1}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{B.6}$$

Now, we use the slightly modified versions of these polynomials, which all have $\phi_n(0) = 0$: $\phi_n(t) = B_n(t) - B_n$. It is easy to show using the generating function that

$$\phi_n(x+1) - \phi_n(x) = nz^{n-1} \tag{B.7}$$

(and hence we obtain a formula for the sum of the kth powers, which is also not what we're after here), and using some more uninteresting manipulation on the generating functions, obtain the following properties of ϕ_n:

$$\phi_n^{(n-k)}(1) = \phi_n^{(n-k)}(0), \qquad \phi_n^{(n-k)} = \frac{n!}{k!} B_k, \tag{B.8}$$

and in particular $\phi_n^{(n-1)}(0) = -n!/2$ and $\phi_n^{(n)}(0) = n!$. Putting this into Darboux's result,

$$\begin{aligned}(z-a)f'(a) =& f(z) - f(a) - \frac{z-a}{2}\left(f'(z) - f'(a)\right) \\ &- \sum_{m=1}^{n-1} \frac{B_{2m}(z-a)^{2m}}{(2m)!} \left(f^{(2m)}(z) - f^{(2m)}(a)\right) \\ &- \frac{(z-a)^{2n+1}}{(2n)!} \int_0^1 \phi_{2n}(t) f^{(2n+1)}(a + (z-a)t)\, dt.\end{aligned} \tag{B.9}$$

Putting $z = a+1$ and writing $F = f'$, this becomes

$$\begin{aligned}\int_a^{a+1} F(x)\, dx =& \frac{1}{2}\left(F(a) + F(a+1)\right) + \\ &+ \sum_{m=1}^{n-1} \frac{B_{2m}}{(2m)!} \left(F^{(2m-1)}(a+1) - F^{(2m-1)}(a)\right) \\ &+ \frac{1}{(2n)!} \int_0^1 \phi_{2n}(t) F^{(2n)}(a+t)\, dt.\end{aligned} \tag{B.10}$$

Writing $a+1, a+2, \dots, b$ instead of a in this formula and summing gives the result.

Obvious application Remember how you proved Stirling's formula in IA Probability? Now we have the equipment, we can derive the full asymptotic series in one go!

Corollary 31 (Stirling's series).

$$\log n! \sim n \log\left(\frac{n}{e}\right) + \frac{1}{2}\log 2\pi n + \sum_{k=2}^{\infty} \frac{(-1)^k B_k}{k(k-1)n^{k-1}}. \tag{B.11}$$

Proof. Take $F(t) = \log t$ in the Euler–Maclaurin formula. □

B.2. The Poisson Summation Formula

Sometimes a series converges too slowly to sum reliably. In the case where the summand has a nice Fourier transform, however, we can use the following result, due to Poisson:

Theorem 32 (Poisson Summation Formula). *Let f be an absolutely integrable function on $\mathbb{R}$. Then*

$$\sum_{n\in\mathbb{Z}} f(x+nT) = \frac{1}{T}\sum_{k\in\mathbb{Z}} \tilde{f}(k/T)e^{2\pi ikx/T}, \tag{B.12}$$

where $\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx$ is the Fourier transform of f.[2]

Proof. This is really simple:[3] since f is absolutely integrable, we can bound the sum on the left by an integral to check it converges. The left-hand side is also periodic with period 1. The right-hand side is a Fourier series expansion, and it is easy to check (by interchanging the sum and the integral) that the Fourier coefficients are what they should be, by using $\sum_n \int_{Tn}^{Tn+T} = \int_{-\infty}^{\infty}$. □

Application: Theta functions Define the basic θ-function as

$$\theta(z;\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi in^2\tau + 2\pi inz\right), \tag{B.13}$$

where $z \in \mathbb{C}$ and τ lives in the upper half-plane: $\Im(\tau) > 0$. It is easy to check that this satisfies the following functional equation:

$$\theta(z+a+b\tau;\tau) = \exp\left(-\pi ib^2\tau - 2\pi ibz\right)\theta(z;\tau), \tag{B.14}$$

and in particular we have that θ is periodic in z with period 1. Now, if we apply the Fourier transform to the summand as a function of n, we obtain

$$\int_{-\infty}^{\infty} e^{-2\pi ikn} \exp\left(\pi in^2\tau + 2\pi inz\right) dn = (-i\tau)^{-1/2}\exp\left(-i\pi(k-z)^2/\tau\right) \tag{B.15}$$

by the usual methods for computing such Fourier transforms. We can now jiggle the term in the exponential about to split off the k:

$$(-i\tau)^{-1/2}\exp\left(-i\pi(k-z)^2/\tau\right) = (-i\tau)^{-1/2}e^{-\pi iz^2/\tau}\exp\left(-\pi ik^2/\tau + 2\pi ik(z/\tau)\right), \tag{B.16}$$

and then we can apply the Poisson Summation Formula to obtain

$$\theta(z;\tau) = \frac{e^{i\pi z^2/\tau}}{\sqrt{i\tau}}\theta\left(\frac{z}{\tau};\frac{i}{\tau}\right); \tag{B.17}$$

this is a really important identity for θ-functions; it is sometimes called *Jacobi's imaginary transformation.*

From our point of view, this result is useful because if τ is such that one series is slow to converge (perhaps it has small imaginary part, for example), then the other side of the equation converges extremely rapidly, so we can always calculate θ-functions efficiently. This is very useful for doing computations with elliptic functions.

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[2]Note the convention!

[3]So simple that it sometimes shows up as a IB exam question…

C. *Langer's Uniform Airy Approximation

The connection formulae may have left you wondering whether there is a *unified* way to approximate a solution across a turning point of q. Indeed there is: Langer found it in 1937.[1] We give a more modern summary.

Starting as usual with

$$\varepsilon^2 y'' = qy, \tag{C.1}$$

we try and find a solution in the form

$$y(x) = A(x)Y(\xi(x)), \tag{C.2}$$

where $Y(\xi)$ satisfies

$$\frac{d^2Y}{d\xi^2} = \mathcal{Q}(\xi)Y = 0; \tag{C.3}$$

for a turning point, a sensible equation to look at will just be $\mathcal{Q}(\xi) = \xi$, but we can run the analysis more generally, so that it would also apply to, for example, a quadratic potential well with the Hermite functions as archtypal solutions.

Comparing the Liouville–Green approximations for these two equations,

$$y(x) = (q(x))^{-1/4} \exp\left(-\frac{1}{\varepsilon}\int_{x_0}^{x} \sqrt{q(x')}\, dx'\right) \tag{C.4}$$

$$Y(\xi) = (\mathcal{Q}(\xi))^{-1/4} \exp\left(-\int_{\xi_0}^{\xi} \sqrt{\mathcal{Q}(\xi')}\, d\xi'\right), \tag{C.5}$$

suggests that perhaps we should define

$$A(x) = \left(\frac{\mathcal{Q}(\xi)}{q(x)}\right)^{1/4} \tag{C.6}$$

$$\int_{\xi_0}^{\xi} \sqrt{\mathcal{Q}(\xi')}\, d\xi' = \frac{1}{\varepsilon}\int_{x_0}^{x} \sqrt{q(x')}\, dx', \tag{C.7}$$

the latter being an implicit equation for ξ. Then

$$\frac{d\xi}{dx} = \frac{1}{\varepsilon}\left(\frac{q(x)}{\mathcal{Q}(\xi)}\right)^{1/2} = \frac{1}{\varepsilon A^2}, \tag{C.8}$$

so if we substitute in the original equation,

$$\begin{aligned}
y''(x) - q(x)y(x) &= \varepsilon^2 \frac{d^2}{dx^2}(A(x)Y(\xi(x))) - q(x)A(x)Y(\xi(x)) && \text{(C.9)}\\
&= \varepsilon^2\varepsilon^2 \frac{A''}{A} AY + \varepsilon^2 \frac{A}{A}\left(2\frac{d\xi}{dx}A' + 2\frac{d^2\xi}{dx^2}A\right)Y' + \varepsilon^2\left(\frac{d\xi}{dx}\right)^2 AY'' - qAY && \text{(C.10)}\\
&= \varepsilon^2\gamma(x)y + 0 + \frac{q}{\mathcal{Q}}A\mathcal{Q}Y - qAY && \text{(C.11)}\\
&= \varepsilon^2\gamma(x)y, && \text{(C.12)}
\end{aligned}$$

[1] Actually, Langer describes a more general version using Bessel functions. See [5]

where the middle term vanishes because $A^2 \frac{d\xi}{dx} = 1/\varepsilon$ is constant, and in the last term we have used the equation $Y'' = \mathcal{Q}Y$. The function γ has expression

$$\gamma(x) = \left(\frac{d\xi}{dx}\right)^{1/2} \frac{d^2}{dx^2} \left(\frac{d\xi}{dx}\right)^{-1/2} = \left(\frac{q(x)}{\mathcal{Q}(\xi)}\right)^{1/4} \frac{d^2}{dx^2} \left(\frac{q(x)}{\mathcal{Q}(\xi)}\right)^{-1/4} \tag{C.13}$$

Then the region of validity of the approximation is detemined by

$$|q(x)| \gg \varepsilon^2 \gamma(x), \tag{C.14}$$

which can be made much wider than a standard JWKB approximation.

For a turning point, wlog at $x = x_0$, we of course choose $Q(\xi) = \xi$, so that the implicit equation for ξ reduces to

$$\int_{\xi_0}^{\xi} \xi'^{1/2} \, d\xi' = \frac{1}{\varepsilon} \int_{x_0}^{x} \sqrt{q(x')} \, dx' \tag{C.15}$$

$$\xi(x) = \left(\frac{3}{2\varepsilon} \int_{x_0}^{x} \sqrt{q(x')} \, dx'\right)^{2/3}, \tag{C.16}$$

and hence

$$y(x) = (\xi(x))^{1/4} (q(x))^{-1/4} \operatorname{Ai}(\xi(x)); \tag{C.17}$$

it can be shown that this function is continuous across the turning point, and fits to the JWKB solution in all three regions.

This can be developed further using other archetypal potential shapes, such as the well or scattering from a barrier. See, for example, [2] § 2.3, whence this exposition is derived.

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