Summary: The Method of Steepest Descent

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1 Method

Suppose we are given the complex integral

$$\int_C f(z)e^{x\phi(z)}\,dz, \tag{1}$$

where C is a contour in the complex plane that may have finite endpoints, or specified directions for large z, $f(z)$ and $\phi(z)$ are analytic, and $x \uparrow \infty$ is real. To apply the Method of Steepest Descent to obtain the asymptotic expansion of this integral:

1. **Find the appropriate contour.**
   a) *Separate $\phi(z)$ into its real and imaginary parts,*

   $$\phi(p + iq) = u(p, q) + iv(p, q). \tag{2}$$

   b) *Find the stationary points of $\phi(z)$ and their natures* (i.e. where $\phi'(z) = 0$, and the smallest nonzero derivative) if any.
   c) *Find the places where $u \to -\infty$ as $|z| = \sqrt{p^2 + q^2} \to \infty$.* These are often called wells or valleys.
   d) *Find the steepest descent contours.* I.e. find the solutions of the equation $v =$ const.
   e) *If the contour C has any finite endpoints a, find all of the steepest descent contours that pass through them.* i.e. solutions to $v(p, q) = v(\Re(a), \Im(a))$.
   f) *Find the steepest descent contours through any stationary points.*
   g) *Deform C into a combination of the above steepest-descent contours using Cauchy's Theorem.* This is the hardest step: the contour needs to be deformed to a set of steepest descent contours that pass through any endpoints, in such a way that the integral is finite throughout the deformation. (This means that any infinite sections remain in the same valleys, although their precise directions can change.) It may also be necessary to include a contour crossing between two valleys that passes through a stationary point. If there are no endpoints, we instead deform the contour to a steepest descent contour through an appropriate stationary point.

2. **Choose the important sections of the contour and parametrise them.**
   a) *Divide the new contour into small sections around each endpoint and stationary point and discard the rest.* The principle here is the same as Laplace's Method: only these small sections will contribute.

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b) *Parametrise each section of contour locally.* If the contours do not have a simple parametrisation (as do straight lines, for example), find functions $p(s), q(s)$ so that $p(0) + iq(0) = z_0$, $v(p(s), q(s)) = v(p(0), q(0))$ to sufficiently high accuracy (this normally means a certain number of terms in the Taylor expansion of the latter equation vanish: for lowest order, the tangent/first-order approximation is sufficient). There are two possible ways of doing this:

- If the steepest-descent contour has a simple parametrisation as a straight line or a circle, it is often easier to parametrise the contour using its standard parametrisation. This results in an integral of the form
$$\int_a^b e^{x\phi(z(s))} f(z(s))\, ds, \tag{3}$$
which may be examined in the same way as a Laplace integral, by expanding $e^{x\phi(z(s))}$ about the stationary point and integrating.

- On the other hand, in the more common case that the contour is difficult to parametrise exactly, it is preferable to use the following method. Since ϕ is analytic and has a stationary point at z_0, say of order p, it is possible to write $\phi(z) = \phi(z_0) - g(z)^p$, where $g(z)/(z - z_0) \to A \neq 0$ as $z \to z_0$ and is nonzero in an open neighbourhood of z_0. (This follows from the definition of a zero of order p and the binomial theorem.) But this means that if we take $s = g(z)$, we immediately reduce ϕ to the form $\phi(z(s)) = \phi(z_0) - s^p$, and will not need to expand the exponential. The approximation comes from finding z in terms of s from $s = g(z)$, which is normally done by substituting a power series for the inverse function into g and equating coefficients in $s = g(z(s))$. The p roots of $-1/A$ give the p different steepest descent contours on which ϕ decreases. An practical way to do this by hand is to look for a power series $F(s) = z_0 + \sum_{k=1}^{\infty} c_k s^k$ so that $\phi(F(s)) = \phi(z_0) - s^p$; such a parametrisation can be shown to exist by a similar proof to above. We can understand this as an analytic transformation that changes the local structure of the particular stationary point at z_0 into one with a prototypical form, which has the real axis as a steepest descent contour and no higher-order changes. To lowest order, this simply means undoing the rotation and scaling that take $-(z - z_0)^p$ to $A(z - z_0)^p$, equivalent to changing variables to make the tangent to the steepest descent contour at z_0 into the real axis. Again there are p different possibilities, coming from the p solutions to $c_1^p \phi^{(p)}(z_0)/p! = -1$, which give p different power series.

3. **Calculate the approximation integral**.

a) *Substitute the parametrisation into the integral.* So $dt = (p'(s) + iq'(s))\, ds$, and since we are on a steepest-descent contour, $\phi(z(s)) = \phi(0) + \big(u(p(s), q(s)) - u(p(0), q(0))\big)$, at least to the same order as the parametrisation.

b) *Expand and evaluate as in Laplace's Method.* Bearing in mind that the accuracy of the parametrisation of the contour limits the accuracy of the result: if in doubt, approximate the contour more accurately and see if the coefficients change. The differential also needs to be expanded as a series.

As usual, the first term is straightforward to find. It only requires a linear approximation to the steepest-descent contour through the point. To produce more terms, we need more information about the contour in the neighbourhood of the point: this amounts to creating a higher-order Taylor expansion for the parametrisation of the contour at the point.

Alternative approach It is not necessary to calculate the contour as we have described above, essentially determining a local inverse to $\phi(z)$. Instead we can take the tangent approximation, take a small chunk of this contour as approximating the whole steepest-descent contour, substitute the corresponding change of variables into the function, and then proceed as in Laplace's Method, expanding ϕ near the point, substituting to remove the leading x behaviour, and then expanding the exponential and taking the limits of the integral to $\pm\infty$.

This works because a if we take a sufficiently small interval of the tangent, the difference between this integral and the integral along a corresponding portion of the true steepest-descent contour is exponentially small compared to the integrals themselves. For example, suppose we are looking at a stationary point; by Cauchy's theorem, the difference is represented by the integral along two arcs joining the steepest-descent contour to the tangent, and on these arcs, the maximum of the real part of ϕ is strictly smaller than ϕ at the stationary point. Thus the integrals along the arcs are bounded by $2\pi\varepsilon e^{x\Re\phi(z_1)}$ for some z_1 with $|z_1 - z_0| = \varepsilon$, and this is exponentially smaller than $e^{x\Re\phi(z_0)}$, essentially because we pass from the tangent to the steepest-descent contour through the region where ϕ is smaller than $\phi(z_0)$.[1] See Figure 1.

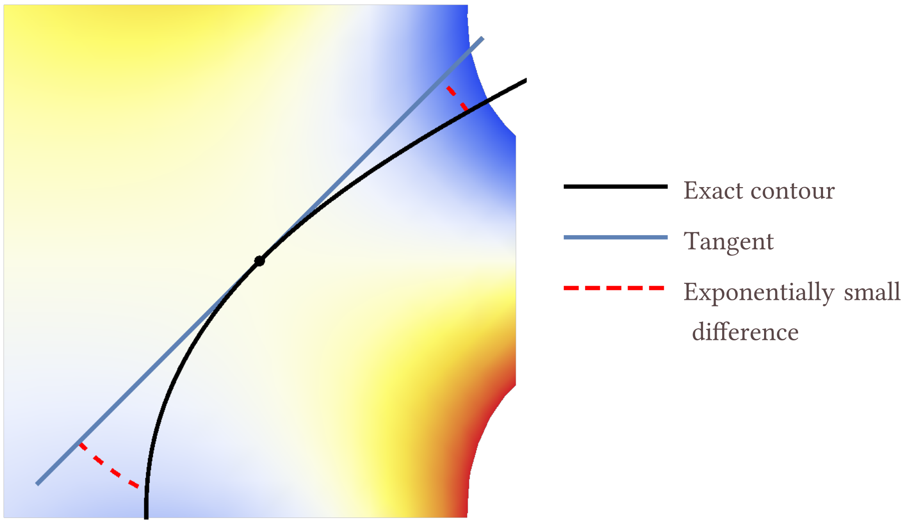


Figure 1: Difference between the integral along a small section of the steepest-descent contour near a stationary point and the integral along a small section of its tangent at the point. The red arcs lie entirely in the blue region, below the value at the stationary point, and hence their integrals' values are exponentially smaller than the values of the other two integrals for large x.

The advantages of this method are that we do not have to calculate anything more than the tangent to the steepest-descent contour before expanding, and that the exponential is normally regarded as easier to expand than a general power series.

A disadvantage is that the limits have to be treated with even more care than usual, since it is easy to take them to infinity at the wrong time, and accidentally write down a divergent integral. Moreover, we have also not actually simplified the calculation: the local inverse method has more work in finding the contour expansion, but once this is done, the exponential only contains one term, the derivative of the change of variables providing the series expansion. Only one term in this series contributes to each term in the asymptotic expansion. On the other hand, in the tangent method, the substitution is simple, but the exponential has to be expanded; moreover, several terms from this expansion contribute to each term in the asymptotic expansion.

As so often in this subject, the total amount of work remains similar, although it is done in different places. You should try both, and decide which you find easier. We shall use both on our example.

2 Example

This is best illustrated by an example. Consider

$$I(x) = \int_{-\infty}^{\infty} e^{ix(t^4 - 4t)} dt, \tag{4}$$

[1] This is rather heuristic, but one can firm up the calculation, taking more care over the actual region the contour passes through, and the local structure of the stationary point.

so

$$\phi(z) = iz^4 - 4iz, \tag{5}$$

and then if $z = p + iq$ and $\phi = u + iv$,

$$u = -4p^3q + 4pq^3 + 4q, \qquad v = p^4 + q^4 - 6p^2q^2 - 4p. \tag{6}$$

Next, the structure of ϕ for large $|z|$: writing $z = re^{i\theta}$, the dominant term is $iz^4 = r^4 \exp i(4\theta + \pi/2)$, and the real part of this is negative when $\cos(4\theta + \pi/2) < 0$, or $\sin 4\theta > 0$, so θ lies in $(0, \pi/4) \cup (\pi/2, 3\pi/4) \cup (\pi, 5\pi/4) \cup (3\pi/2, 7\pi/4)$.

Therefore the contour should have its ends in the regions with $0 < \arg z < \pi/4$ and $-\pi < \arg z < -3\pi/4$, as these are the only regions where the integrand tends to 0 as $|z| \to \infty$ that we can deform the ends of the original contour into without passing over a region where the integrand (and hence the integral) diverges as $|z| \to \infty$. See Figure 2.

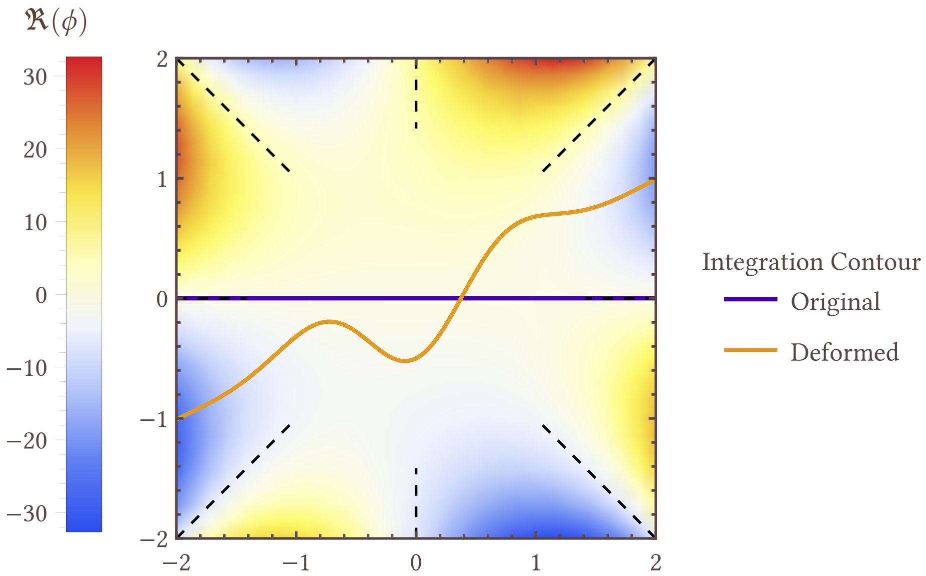


Figure 2: Initial contour, regions where $\Re(\phi) \to \pm\infty$, and first deformed contour.

Next, we have to understand where the stationary points are and find the steepest descent contours through these points. We have

$$\phi'(z) = 4i(z^3 - 1), \tag{7}$$

so the stationary points are $z_0 = 1$, $z_1 = e^{2\pi i/3}$ and $z_2 = e^{-2\pi i/3}$. We also have

$$\phi(z_0) = -3i, \qquad \phi(z_1) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i, \qquad \phi(z_2) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i. \tag{8}$$

It is important to know which ones out of these the contour will pass through, and the only way to find out is to examine the steepest descent contours, namely

$$v = p^4 + q^4 - 6p^2q^2 - 4p = C \tag{9}$$

For z_0, $C = -3$, while for both z_1 and z_2, $C = 3/2$. Determining the shape of these two sets of curves is the difficult bit. The equation is quadratic in q^2, so at least we can find an exact set of solutions, namely

$$q = \pm\sqrt{3p^2 \pm \sqrt{8p^4 + 4p + C}}. \tag{10}$$

In general, each curve will have its ends in two different regions, and pass through a certain number of stationary points, where it crosses over others depending on the order of the stationary point: a stationary point of order k has k crossings. (Closed curves cannot occur for analytic functions by the maximum principle: if a harmonic function is constant on the boundary of a compact region, it is constant inside.) If it does not pass through a stationary point, it must connect regions with opposite signs of $\Re(\phi)$. Here every stationary point is simple, so only two curves cross at each. Another useful piece of data is sometimes provided by considering a value of C for which the curves are easier to draw: the curves for the value we are interested in cannot cross these.

In this case we know that all of the curves are asymptotic to $p^4 - 6p^2q^2 + q^4 = 0$, (see Figure 3) so we have to find out how each of these joins to the others for the values of C we are interested in.

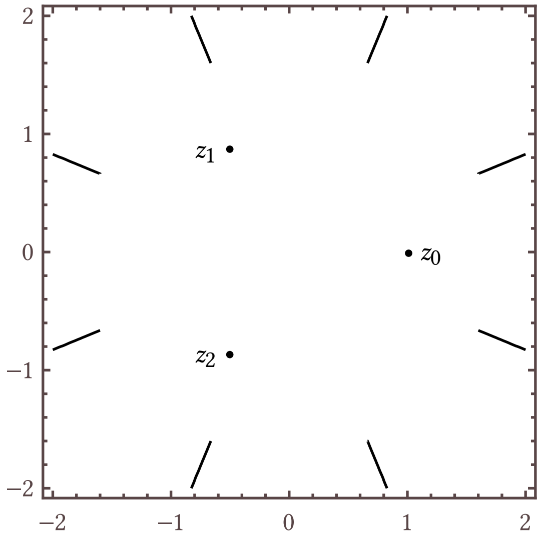


Figure 3: Stationary points of ϕ, and approximate asymptotes $p^4 - 6p^2q^2 + q^4 = 0$ of all steepest descent contours.

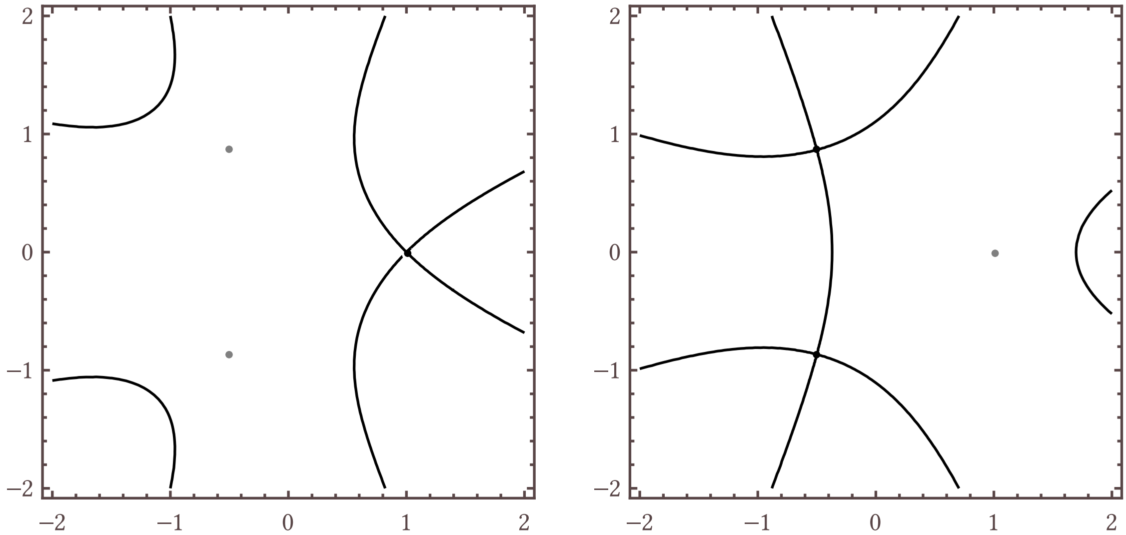
If $C < 0$, as it is for z_0, we can see that the curves cannot cross $p = 0$. For $C = -3$, the local structure at z_0 implies that we have two curves crossing, at angle $\pm\frac{1}{2}\arg i = \pm\pi/4$ to the p-axis, and we end up with the situation in Figure 4a (the curves with negative p are not important, but drawn for completeness). You can test for yourself that there is no other way of joining the asymptotic curves from Figure 3 together that is consistent with this information.

For $C > 0$, two curves do cross the q-axis once, and because they don't cross the $C = -3$ curves, we know which octants they lie in. These also contain z_1 and z_2. The remaining octants are on the left, and they must be joined by a curve that intersects these two curves in z_1 and z_2. (There is also a curve to the right of z_0, but again it is not important.) Putting this information together gives Figure 4b.

We now choose the new integration contour: we know where it starts and ends, and we must pass through z_2 to escape the region $-\pi < \arg z < -3\pi/4$, and z_0 to escape $0 < \pi < \pi/4$. Also, since z_0 and z_2 have different values of v, we must go off to infinity between them in order to change steepest descent contour. Therefore our only choice is the contour in Figure 5.

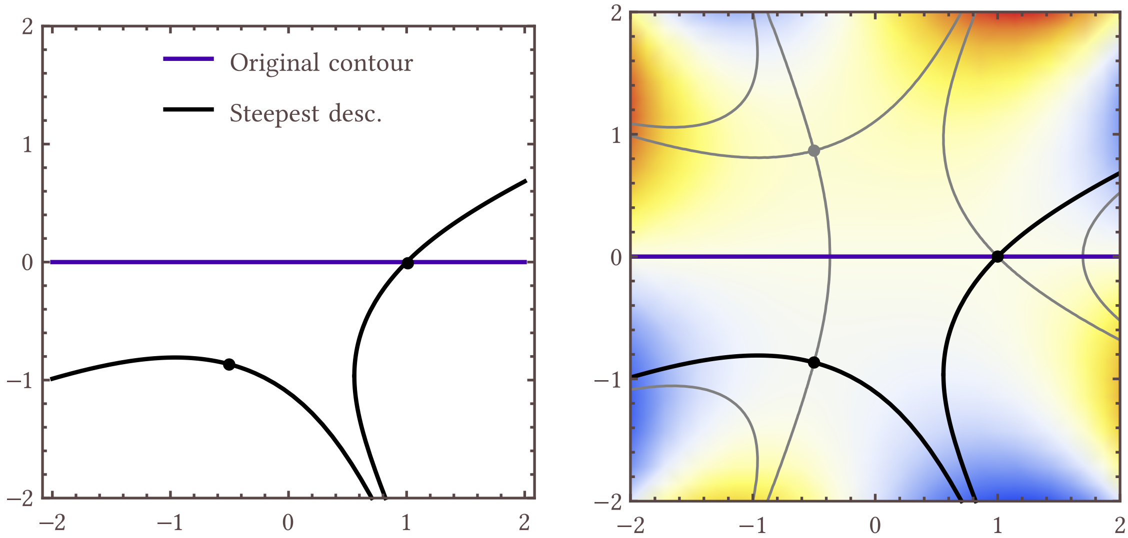
Notably, even though z_1 is a stationary point, it does not lie on the new integration contour and hence plays no part in the analysis. If we had started with a different contour, it could have been on the steepest descent contour and made the dominant contribution. But it isn't, so it doesn't.

We now need to find the contributions from z_0 and z_1. Firstly, we look at the lowest order to decide which is the dominant contribution: in this case, we expect the contribution from z_0 to be $\propto x^{-1/2}e^{x\Re(\phi(z_0))} = x^{-1/2}e^0$, while that from z_1 is $\propto x^{-1/2}e^{x\Re(\phi(z_1))} = x^{-1/2}e^{-3\sqrt{3}x/2}$. The latter is exponentially small compared to the former, so it is only necessary to find the expansion about $z_0 = 1$.



(a) $\Im(\phi) = -3$ (b) $\Im(\phi) = 3/2$

Figure 4: Steepest descent contours through the stationary points



(a) Initial and final contours (b) All information together

Figure 5: Appropriate steepest descent contour for integral

As we noted previously, we have two approaches to this.

Local steepest-descent contour method To do this, we need to know the local structure of the contour: in other words, we want a parametrisation of the contour that we can expand in a power series, so we can do the usual Laplace's method idea.

Since the contour does not have a simple parametrisation, we look for *complex* coefficients c_n so that $F_n(t) = \sum_{k=0}^{n} c_k t^k$ satisfies $\phi(F_n(t)) = \phi(z_0) - t^2 + O(t^n)$ and the we have the correct tangent . In this case, we find

$$\phi(1 + c_1 t + \cdots) = -3i + 6ic_1^2 t^2 + 4i(c_1^3 + 3c_2 c_1)t^3 + i(c_1^4 + 12c_2 c_1^2 + 12c_3 c_1 + 6c_2^2)t^4 + \cdots \tag{11}$$

and equate coefficients with $-3i - t^2$: this method has rather simpler expressions than the corresponding procedure using u and v with two sets of real coefficients. (Feel free to investigate this yourself.) Solving these equations and choosing the root $c_1 = e^{i\pi/4}/\sqrt{6}$, we find that

$$z = 1 + \frac{1+i}{2\sqrt{3}}t - \frac{i}{18}t^2 - \frac{7(1-i)}{432\sqrt{3}}t^3 + \frac{7}{1944}t^4 - \frac{77(1+i)}{62208\sqrt{3}}t^5 + O(t^6) \tag{12}$$

satisfies

$$\phi(z) = -3i - t^2 + O(t^6). \tag{13}$$

The successive approximations we find to the contour by this process are shown in Figure 6. We now know that the c_i exist, but we shall continue to write them as unevaluated symbols to save space, and to make the calculation clearer.

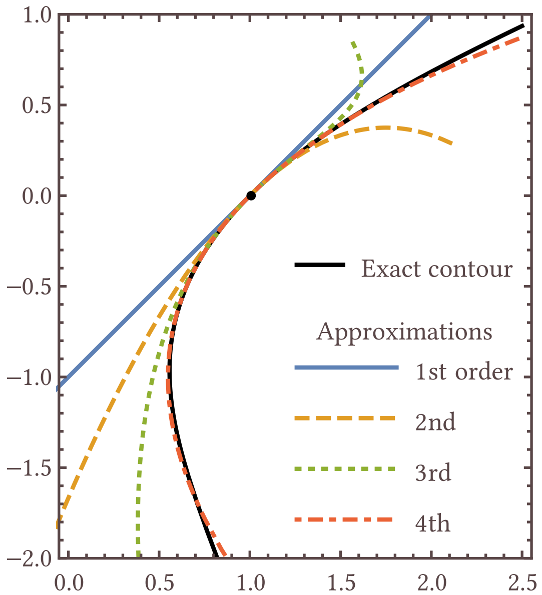


Figure 6: Successive approximations to the steepest descent contour near the dominant stationary point $z_0 = 1$. Only the odd-order ones add to the order of approximation (the even ones vanish when differentiated and integrated).

Now we simply substitute in the integral: if the steepest-descent contour is γ, and γ_ε the segment parametrised by $-\varepsilon < t < \varepsilon$, we have

$$I(x) = \int_{-\infty}^{\infty} e^{x\phi(t)}\, dt = \int_{\gamma} e^{x\phi(z)}\, dz \sim \int_{\gamma_\varepsilon} e^{x\phi(z)}\, dz = \int_{-\varepsilon}^{\varepsilon} e^{x\phi(z(t))} z'(t)\, dt,$$

and the measure changes as

$$dz = z'(t)\, dt = (c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + 6c_6 t^5 + O(t^6))\, dt \tag{14}$$

so we have

$$I(x) \sim e^{-3ix} \int_{-\varepsilon}^{\varepsilon} e^{-xt^2} (c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + 6c_6 t^5 + O(t^6))\, dt. \tag{15}$$

This is now just like a Laplace integral: changing variables to $u = \sqrt{x}t$ gives

$$I(x) \sim \frac{e^{-3ix}}{\sqrt{x}} \int_{-\sqrt{x}\varepsilon}^{\sqrt{x}\varepsilon} e^{-u^2} \left(c_1 + 2c_2 \frac{u}{\sqrt{x}} + 3c_3 \frac{u^2}{x} + 4c_4 \frac{u^3}{x^{3/2}} + 5c_5 \frac{u^4}{x^2} + 6c_6 \frac{u^5}{x^{5/2}} + O\left(\frac{u^6}{x^3} \right) \right) du \tag{16}$$

$$= \frac{e^{-3ix}}{\sqrt{x}} \left(\int_{-\sqrt{x}\varepsilon}^{\sqrt{x}\varepsilon} e^{-u^2} \left(c_1 + 2c_2 \frac{u}{\sqrt{x}} + 3c_3 \frac{u^2}{x} + 4c_4 \frac{u^3}{x^{3/2}} + 5c_5 \frac{u^4}{x^2} + 6c_6 \frac{u^5}{x^{5/2}} \right) du + O\left(\frac{1}{x^3} \right) \right) \tag{17}$$

$$= \frac{e^{-3ix}}{\sqrt{x}} \left(\int_{-\infty}^{\infty} e^{-u^2} \left(c_1 + 2c_2 \frac{u}{\sqrt{x}} + 3c_3 \frac{u^2}{x} + 4c_4 \frac{u^3}{x^{3/2}} + 5c_5 \frac{u^4}{x^2} + 6c_6 \frac{u^5}{x^{5/2}} \right) du + O\left(\frac{1}{x^3} \right) \right), \tag{18}$$

where we have applied the usual procedure of extracting the remainder term from the integral using the exponential suppression and replacing the limits by $\pm\infty$. The odd terms vanish, and we apply the usual integral results to find

$$\int_{-\infty}^{\infty} e^{ix(t^4-4t)}\, dt = e^{-3ix+i\pi/4} \frac{\sqrt{\pi}}{\sqrt{6x}} \left(1 + \frac{7i}{144x} - \frac{385}{41472x^2} + O\left(\frac{1}{x^3} \right) \right). \tag{19}$$

(The odd terms may not vanish if there is an extra function in the integral.)

Tangent method The expansion of ϕ at the stationary point begins

$$\phi(z) = -3i + 6i(z-1)^2 + O((z-1)^3). \tag{20}$$

Substituting $z = 1 + e^{i\theta}u$, this becomes $-3i + 6ie^{2i\theta}u^2 + O(u^3)$. The steepest descent directions occur when the coefficient of u^2 is real and negative, so we need $2\theta + \pi/4 = (2k+1)\pi$. There are two choices, $\theta = \pi/4$ and $\theta = 5\pi/4$, with corresponding coefficients $\pm e^{i\pi/4}$, but the contour passes from left to right, so we choose the first of these. So we take $z = 1 + e^{i\pi/4}u$. Substituting this into ϕ gives

$$\phi(1 + e^{i\pi/4}u) = -3i - 6u^2 + 4e^{-i\pi/4}u^3 - iu^4, \tag{21}$$

so $I(x)$ is exponentially close to

$$\int_{-\varepsilon}^{\varepsilon} \exp\left(x(-3i - 6u^2 + 4e^{-i\pi/4}u^3 - iu^4) \right) e^{i\pi/4} du = e^{-3ix+i\pi/4} \int_{-\varepsilon}^{\varepsilon} \exp\left(-6xu^2 + 4xe^{-i\pi/4}u^3 - ixu^4 \right) du. \tag{22}$$

We now proceed as in Laplace's Method: to remove the x-dependence from the leading term, we put $u = v/\sqrt{6x}$, and so the integral becomes

$$\frac{e^{-3ix+i\pi/4}}{\sqrt{6x}} \int_{-\varepsilon\sqrt{6x}}^{\varepsilon\sqrt{6x}} \exp\left(-v^2 + \frac{\sqrt{2}e^{-i\pi/4}}{3\sqrt{3}} \frac{v^3}{x^{1/2}} - \frac{i}{36} \frac{v^4}{x} \right) du. \tag{23}$$

We must now expand the exponential to the required order in x. We shall go as far as showing a remainder of $O(x^{-3})$. Writing the coefficients as $C_1 = \frac{\sqrt{2}e^{-i\pi/4}}{3\sqrt{3}}$ and $C_2 = -\frac{i}{36}$, we find

$$\begin{aligned} \exp\left(C_1 \frac{v^3}{x^{1/2}} + C_2 \frac{v^4}{x} \right) = 1 + C_1 \frac{v^3}{x^{1/2}} &+ \left(C_2 v^4 + \frac{1}{2} C_1^2 v^6 \right) \frac{1}{x} + \left(C_1 C_2 v^7 + \frac{1}{6} C_1^3 v^9 \right) \frac{1}{x^{3/2}} \\ &+ \left(\frac{1}{2} C_2^2 v^8 + \frac{1}{2} C_1^2 C_2 v^{10} + \frac{1}{24} C_1^4 v^{12} \right) \frac{1}{x^2} \\ &+ \left(\frac{1}{2} C_1 C_2^2 v^{11} + \frac{1}{6} C_1^3 C_2 v^{13} + \frac{1}{120} C_1^5 v^{15} \right) \frac{1}{x^{3/2}} + O\left(\frac{v^{12} + v^{18}}{x^3} \right) \end{aligned} \tag{24}$$

Inserting this into the integral, we can split the integral into a finite sum of integrals, and the order term. Careful choice of ε allows us to replace the limits by $\pm\infty$ with a small enough error, and we find that

$$I(x) = \frac{e^{-3ix+i\pi/4}}{\sqrt{6x}} \int_{-\infty}^{\infty} e^{-v^2}\left(1 + \left(C_2 v^4 + \frac{1}{2}C_1^2 v^6\right)\frac{1}{x} + \left(\frac{1}{2}C_2^2 v^8 + \frac{1}{2}C_1^2 C_2 v^{10} + \frac{1}{24}C_1^4 v^{12}\right)\frac{1}{x^2}\right) dv + O(x^{-3}), \tag{25}$$

the terms with odd powers of v being zero by symmetry. This is a sum of integrals we know how to compute, and plodding through the algebra again leads us to (19).

Accuracy We can see the impressive accuracy of the nth order approximation, even for small n and quite small x, in Figure 7, which shows the approximations

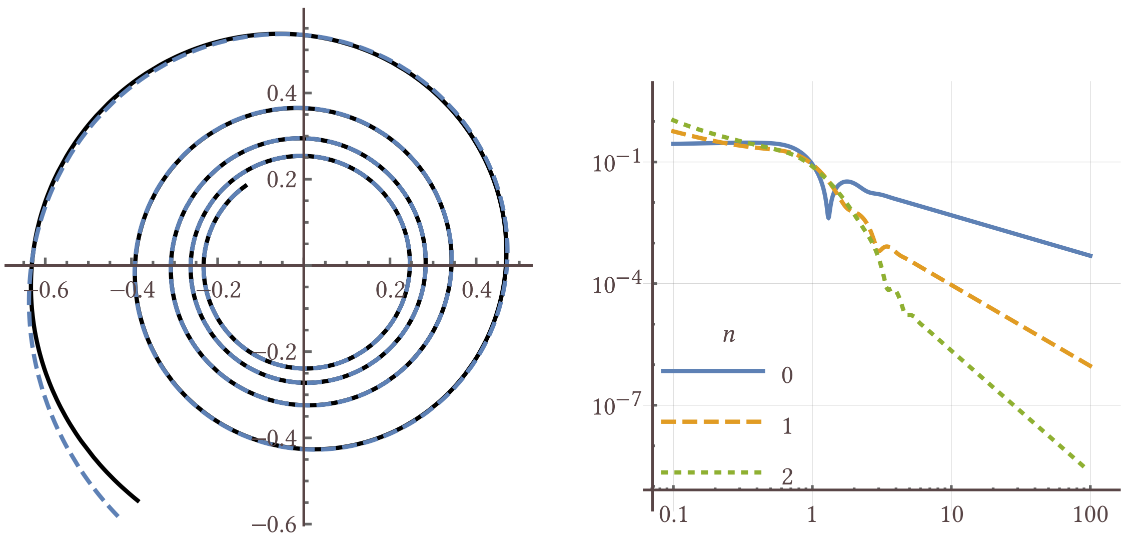
$$A_0(x) = e^{-3ix+i\pi/4}\frac{\sqrt{\pi}}{\sqrt{6x}}, \qquad A_1(x) = e^{-3ix+i\pi/4}\frac{\sqrt{\pi}}{\sqrt{6x}}\left(1 + \frac{7i}{144x}\right),$$
$$A_2(x) = e^{-3ix+i\pi/4}\frac{\sqrt{\pi}}{\sqrt{6x}}\left(1 + \frac{7i}{144x} - \frac{385}{41472x^2}\right), \tag{26}$$

comparing the $A_0(x)$ to $I(x)$ in (a), and the relative error in all three in (b).

Exercise Carry out the same analysis for

(a) $\displaystyle\int_0^\infty e^{ix(t^4-4t)}\,dt$, finding the first four nonzero terms.

(b) $\displaystyle\int_{-\infty}^\infty \frac{e^{ix(t^4-4t)}}{1+t^2}\,dt$, finding the first two nonzero terms.



(a) Plot of the $I(x)$ (black, solid) in the complex plane with lowest-order approximation $A_0(x)$ (blue, dashed), for x between 1 (bottom left) and 10 (centre)

(b) Log-log plot of the relative error in each approximation, $|A_n(x)/I(x) - 1|$.

Figure 7: The impressive accuracy of the approximation