Harmonic Functions and Conformal Maps

Richard Chapling*

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Setup Let $D, E \subset \mathbb{C}$ be conformally equivalent simply-connected domains, and $g : D \to E$ the conformal equivalence relating them (in particular, g is holomorphic and a bijection). Given a real-valued harmonic function ϕ on D, we would like to find one on E.

Nitpicking Sadly, we can't *quite* do this. Firstly, the harmonic function is a function $\mathbb{R}^2 \to \mathbb{R}$, and D, E are subsets of $\mathbb{C}$. Therefore we first define some isomorphisms between $\mathbb{R}^2$ and $\mathbb{C}$, just to tidy everything up and make it correctly defined:

$$c : \mathbb{R}^2 \to \mathbb{C} \qquad\qquad r : \mathbb{C} \to \mathbb{R}^2 \tag{1}$$

$$(x, y) \mapsto x + iy \qquad\qquad z \mapsto \left(\tfrac{1}{2}(z + z^*), \tfrac{1}{2i}(z - z^*)\right) \tag{2}$$

It is clear that these respect the topology and vector-space structure of $\mathbb{R}^2$ and $\mathbb{C}$, and we have

$$c \circ r = \mathrm{Id}_{\mathbb{C}}, \qquad\qquad r \circ c = \mathrm{Id}_{\mathbb{R}^2}; \tag{3}$$

hence they are isomorphisms (we could call c the "complexification" and r the "reification", but that is in conflict with the usual meanings of these words, unfortunately). Therefore we can define the domains

$$\underline{D} = r(D) \subset \mathbb{R}^2, \qquad\qquad \underline{E} = r(E) \subset \mathbb{R}^2, \tag{4}$$

and we now have a sensible domain for ϕ, and can say

$$\phi : \underline{D} \to \mathbb{R}, \tag{5}$$

and now we want to find a harmonic function $\underline{E} \to \mathbb{R}$.[1]

The Play Let's make a map (in the cartographic sense) of what we have so far. We'll do this in the form of a diagram:

$$\begin{array}{ccccc} E & \overset{g}{\underset{\sim}{\longleftarrow}} & D & & \\ c\uparrow\wr\downarrow r & & c\uparrow\wr\downarrow r & & \\ \underline{E} & & \underline{D} & \underset{\phi}{\longrightarrow} & \mathbb{R} \end{array}$$

* Trinity College, Cambridge

[1] Obviously c and r remain bijections onto their images when restricted to D or E, so they are still isomorphisms $D \leftrightarrow \underline{D}$, $E \leftrightarrow \underline{E}$.

The rather strange shape is because I've put all the complex stuff in the top line, the real stuff in the bottom line, and then joined everything up *in the only possible way*: no other combination of maps will join these sets together. This is why such diagrams are useful to us:[2] we can write down constructions like this basically automatically.

Are we done? Well, no: we don't know that the function that is the composition of all of these functions is harmonic, which is the whole point! We could try and check this directly, but let's use some facts from the course:

1. The composition of analytic functions is analytic.

2. The real part of an analytic function is harmonic. In our current language, we can specify this more precisely as: if $h : C \to \mathbb{C}$ is analytic, then $\Re \circ f \circ c : r(C) = \underline{C} \to \mathbb{R}$ is harmonic.

A sensible thing to do would therefore be to extend the above diagram to include an analytic function f of which ϕ is the real part:

$$\begin{array}{ccccc} E & \xleftarrow[\sim]{g} & D & \xrightarrow{f} & \mathbb{C} \\ c \uparrow \wr \downarrow r & & c \uparrow \wr \downarrow r & & \downarrow \Re \\ \underline{E} & \xleftarrow[\underline{g}]{} & \underline{D} & \xrightarrow[\phi]{} & \mathbb{R} \end{array}$$

(notice that the left square is entirely related to the maps between the domains, the right square is about ϕ and f). We would then have, using the first result mentioned above, that $f \circ g^{-1}$ is analytic on E, and hence $\phi \circ \underline{g}^{-1} = \Re \circ (f \circ g^{-1}) \circ c$ is harmonic on $\underline{E}$.

We need to construct f and $\underline{g}$ so that this diagram *commutes* (this means that whichever valid path I take through it between two sets, the answer is the same).[3] $\underline{g}$ is straightforward: we can just define it as $r \circ g \circ c$; it is an isomorphism because it is a composition of isomorphisms.

Therefore the only thing we have left to do is sort f out: we have to find f so that

$$\phi = \Re \circ f \circ c, \quad \text{or} \quad \phi(x, y) = \Re(f(x + iy)). \tag{6}$$

Luckily, there's a clever trick to this: define

$$h(x, y) = \partial_x \phi(x, y) - i \partial_y \phi(x, y). \tag{7}$$

Then the surprising fact is that $H = h \circ r$ is analytic. Since h is clearly continuous with continuous partial derivatives, it suffices to check that h satisfies the Cauchy–Riemann equations: $\Re(h)_x - \Im(h)_y = \phi_{xx} + \phi_{yy} = 0$ since ϕ is harmonic, and $\Re(h)_y + \Im(h)_x = \phi_{xy} - \phi_{yx} = 0$ since the mixed partial derivatives are equal. Hence H is analytic, and we can attempt to define f by integration: set

$$f(z) = H(z_0) + \int_{z_0}^{z} H(w)\, dw, \tag{8}$$

where the path is contained within D: the integral is independent of the choice of path because D is simply connected, and we proved earlier that it is analytic, so we just have to check that $\Re \circ H = \phi$. This is easy to check using the Fundamental Theorem of Calculus for Line Integrals:

$$\begin{aligned} \Re(f(x + iy)) - \phi(x_0, y_0) &= \Re \int_{x_0 + iy_0}^{x + iy} H(w)\, dw = \Re \int_{(x_0, y_0)}^{(x, y)} (\phi_x - i\phi_y)(dx + i\, dy) \\ &= \int_{(x_0, y_0)}^{(x, y)} (\phi_x\, dx + \phi_y\, dy) = \int_{(x_0, y_0)}^{(x, y)} d\phi = \phi(x, y) - \phi(x_0, y_0). \end{aligned}$$

Hence $\phi \circ \underline{g}^{-1}$ is the required harmonic function on $\underline{E}$.

[2] even if we're not category theorists

[3] It's the mathematical version of a vector field being conservative, I suppose.