Hardy's proof of Hardy's Theorem

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> *Zu den bedeutendsten Fortschritten der Mathematik aus der letzten Zeit gehört die Note von Herrn G. H. Hardy* Sur les zéros de la fonction $\zeta(s)$ de Riemann.
>
> E. Landau[1]

We shall provide some comfort by proving the first of a series of results concerning the number of zeros on the critical line. The original proof, due to Hardy, is so simple, ingenious and neat that we shall effectively fill in the details in his brilliant paper, [2]; this proof is in the Bradman class.[2]

Theorem 1 (Hardy). *There are infinitely many zeros of $\zeta(s)$ on $\sigma = \frac{1}{2}$.*

We begin with some preliminary calculation of integrals:[3] consider the contour integral

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(u)y^{-u}\,du,$$

where $\Re(y) > 0$, and $k > 0$, so the contour passes to the right of all the poles of the Gamma--function. Write $s = k + i\tau$. To check it converges,

$$|\Gamma(2+u)| = \left|\int_0^\infty t^{1+u}e^{-t}\,dt\right| \leqslant \left|\int_0^\infty t^{1+k}e^{-t}\,dt\right| = |\Gamma(2+k)|,$$

$$\begin{aligned}\left|\int_{k-i\infty}^{k+i\infty} \Gamma(u)y^{-u}\,du\right| &= \left|\int_{k-i\infty}^{k+i\infty} \frac{\Gamma(2+u)}{u+1}\frac{y^{-u}}{u}\,du\right| \\ &\leqslant y^{-k}\Gamma(2+k)\left(\int_{k-i\infty}^{k+i\infty}\frac{|du|}{|u+1|^2}\right)^{1/2}\left(\int_{k-i\infty}^{k+i\infty}\frac{|du|}{|u|^2}\right)^{1/2} \\ &\leqslant y^{-k}\Gamma(2+k)\int_{-\infty}^{\infty}\frac{d\tau}{k^2+\tau^2} = \frac{\pi}{k}y^{-k}\Gamma(k+2),\end{aligned}$$

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[1] 'One of the most remarkable achievements of recent Mathematics is the note of G. H. Hardy, *On the zeros of Riemann's function* $\zeta(s)$.', [3] (this paper extends the result to L-functions, among other things.

[2] And if you don't understand that reference, you need to read a biographical work on G. H. Hardy.

[3] In a paper by Hardy, how else could it possibly commence?

where Cauchy-Schwarz has been used, along with $|u| < |u+1|$ for $k > 0$.

Therefore we can use Cauchy's theorem to evaluate the integral: it is easy to show that if k is negative and nonintegral, we can obtain a similar bound.[4] The behaviour of $y^{-k}\Gamma(k)$ as $k \to -\infty$ through nonintegral values is easy enough: using the fundamental relation n times gives

$$y^{-k}\Gamma(k) = \frac{y}{k}\frac{y}{k+1}\cdots\frac{y}{k+n}y^{-k-n}\Gamma(k+n),$$

and for $1-k > n > -k$ we see that this tends to zero as $k \to \infty$, since eventually $y < -k$ and the factors become < 1. Therefore we integrate $\Gamma(u)y^{-u}$ around a rectangle with corners $c \pm iR$ and $-N - \frac{1}{2} + iR$ and take $N, R \to \infty$. The Riemann-Lebesgue lemma deals with the top and bottom edges, we have just shown that the left edge has contribution tending to zero, and we conclude that

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(u)y^{-u}\,du = \sum_{\Re(P)<0} \operatorname{Res}(P),$$

where P are the poles of $\Gamma(u)$. Taking $z \to -n$ in the equation

$$\Gamma(z) = \Gamma(z+n+1)\prod_{r=0}^{n}\frac{1}{z+r}$$

shows that the poles are at $z = -n$ with residues $(-1)^n/n!$, for $n = 0, 1, 2, \ldots$. Hence

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(u)y^{-u}\,du = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}y^n = e^{-y}.$$

Perhaps astonishingly, to convince the audience that this is true is easily one of the hardest bits of the proof.[5]

The next part, as they say, is just algebra.

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2 y} = 1 + \frac{1}{\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(u)y^{-u}\sum_{n=1}^{\infty} n^{-2u}\,du = 1 + \frac{1}{\pi i}\int_{k-i\infty}^{k+i\infty} \Gamma(u)y^{-u}\zeta(2u)\,du,$$

which is valid for $k > \frac{1}{2}$, the interchange of sum and integral being easily justified by absolute convergence. Move the integration contour left to the line $k = \frac{1}{4}$. To do so we must include $2\pi i$ of the residue at $u = \frac{1}{2}$, which is $\frac{1}{2}(\pi i)^{-1}\sqrt{\pi/y}$:

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2 y} = 1 + \sqrt{\frac{\pi}{y}} + \frac{1}{\pi i}\int_{1/4-i\infty}^{1/4+i\infty} \Gamma(u)y^{-u}\zeta(2u)\,du.$$

Replacing $\Gamma(u)\zeta(2u) = \pi^u\xi(2u)u^{-1}(2u-1)^{-1}$,

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2 y} = 1 + \sqrt{\frac{\pi}{y}} + \frac{2}{\pi i}\int_{1/4-i\infty}^{1/4+i\infty} \pi^u y^{-u}\frac{\xi(2u)}{2u(2u-1)}\,du.$$

[4] Note that $|\Gamma(u)| < |\Gamma(k)|$ is always true, which can be seen using the fundamental relation, bounding, using $|z| > |\Re(z)|$ and recombining.

[5] Although given that the rest of the proof is a grand total of two pages in the original article, one may expect to be hard-put to find many other bits.

Changing variables with $2u = \frac{1}{4} + it$ and replacing $\xi(1/2+it) = \Xi(t)$, which is real for real t, we obtain after a little more algebra

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2 y} = 1 + \sqrt{\frac{\pi}{y}} + \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\pi}{y}\right)^{1/4+it/2}\frac{\Xi(t)}{\frac{1}{4}+t^2}\,dt,$$

and further, since $\Xi(2t)$ is even ($\zeta(\overline{s}) = \overline{\zeta(s)}$ etc.), we can split the integral at 0 and write it as one on $(0,\infty)$. Replacing $y = \pi e^{2ia}$ with $|a| < \frac{1}{4}\pi$, we obtain

$$\frac{2}{\pi}\int_0^{\infty}\cosh at\,\frac{\Xi(t)}{\frac{1}{4}+t^2}\,dt = 2\cos\tfrac{1}{2}a - e^{ia/2}\left(1 + 2\sum_{n=1}^{\infty}\exp\left(-n^2\pi e^{2ia}\right)\right) \tag{0.1}$$

Remark 2. The first two terms here are invariant under $a \to -a$, so we obtain the following surprising identity:

$$1 + 2\sum_{n=1}^{\infty}\exp\left(-n^2\pi e^{2ia}\right) = e^{-ia}\left(1 + 2\sum_{n=1}^{\infty}\exp\left(-n^2\pi e^{-2ia}\right)\right).$$

If we set $e^{2ia} = -i\tau$, (so $-i\tau = \pi y$, and τ lives in the upper half-plane) we find

$$\theta(\tau) = (-i\tau)^{-1/2}\theta(-1/\tau),$$

where

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau}$$

and the sign of the square root is that which assigns $1^{1/2} = 1$. This is one of the famous *Jacobi theta-functions*, and the formula just derived is its most important identity.[6] Their importance and beauty can hardly be overstated.

Having reached this point, all we need now are a couple of lemmata:

Lemma 3.

$$\lim_{a\to\frac{1}{4}\pi}\int_0^{\infty} t^{2n}\cosh at\,\frac{\Xi(t)}{\frac{1}{4}+t^2}\,dt = \frac{(-1)^n\pi}{4^n}\cos\tfrac{1}{8}\pi \tag{0.2}$$

Proof. We first prove that the integral in (0.1) can be differentiated for $|a| < \frac{1}{4}\pi$. Stirling's formula shows that

$$\log\Gamma(z) \sim (z - \tfrac{1}{2})\log z - z + \log\sqrt{2\pi},$$

and so if $z = \sigma + it$, $t \to \infty$, $\log z \sim \frac{1}{2}i\pi + \log t$, so

$$\log|\Gamma(z)| \sim (\sigma - \tfrac{1}{2})\log t - \tfrac{1}{2}\pi t - \sigma + \log\sqrt{2\pi}.$$

[6]This is probably the least common proof of this identity. Only Titchmarsh appears to use it. It is, admittedly, a rather roundabout and silly way of doing it.

Using this and our previous bounds on $\zeta(s)$, $\Xi(t) = O(t^A e^{-\pi t/4})$, where A is some positive constant. We can therefore differentiate (0.1) as many times as we like with respect to a, giving

$$\int_0^\infty t^{2n} \cosh at \frac{\Xi(t)}{\frac{1}{4}+t^2}\, dt = \frac{(-1)^n \pi}{4^n} \cos \tfrac{1}{2}a - \tfrac{1}{2}\pi \left(\frac{d}{da}\right)^{2n} e^{ia/2} \theta(ie^{2ia}) \qquad (0.3)$$

It remains to show that the second term on the right-hand side of this equation tends to 0. Writing $e^{2ia} = i - \delta$, we find

$$\theta(-1+i\delta) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi(i+\delta)} = \sum_{n=-\infty}^{\infty} (-1)^n e^{-n^2\pi\delta},$$

since n^2 is odd if and only if n is. We can then exploit a trick and write

$$\sum_{n=-\infty}^{\infty} (-1)^n e^{-n^2\pi\delta} = 2\theta(4i\delta) - \theta(2i\delta) = \delta^{-1/2}\left(\theta\left(\frac{i}{4\delta}\right) - \theta\left(\frac{i}{\delta}\right)\right)$$

But the expression on the right is clearly $O(\delta^{-1/2}e^{-\pi/\delta})$ as $\delta \to 0$, as is every one of its derivatives, so $\theta^{(n)}(-1) = 0$ for any integral $n \geqslant 0$. The same is then true for the last term in (0.3), so taking the limit $a \to \frac{1}{4}\pi$ gives the result. $\square$

We can now prove the theorem.

Proof of Hardy's theorem. Suppose that $\zeta(\frac{1}{2}+it)$ has only finitely many real roots. This occurs if and only if $\Xi(t)$ has one sign for $t > T$, say, which we may take to be positive.[7] Then the previous lemma shows that

$$\lim_{a\to\frac{1}{4}\pi} \int_T^\infty t^{2n} \cosh at \frac{\Xi(t)}{\frac{1}{4}+t^2}\, dt = L,$$

say. Hence,

$$\int_T^{T'} t^{2n} \cosh at \frac{\Xi(t)}{\frac{1}{4}+t^2}\, dt \leqslant L$$

for $a < \frac{1}{4}\pi$ and any $T' > T$. Taking $a \to \frac{1}{4}\pi$ in this inequality shows that the integral

$$\int_0^\infty t^{2n} \cosh \tfrac{1}{4}\pi t \frac{\Xi(t)}{\frac{1}{4}+t^2}\, dt$$

is convergent, and hence the left-hand side of (0.1) is uniformly convergent for $|a| < \frac{1}{4}\pi$, so the derivative may be taken at $\frac{1}{4}\pi$ to give

$$\int_0^\infty t^{2n} \cosh \tfrac{1}{4}\pi t \frac{\Xi(t)}{\frac{1}{4}+t^2}\, dt = \frac{(-1)^n\pi}{4^n} \cos \tfrac{1}{8}\pi.$$

Take n now odd, so the right-hand side is negative. Hence in the integral above,

$$\int_T^\infty < \int_0^T < K \int_0^T t^{2n}\, dt = KT^{2n},$$

[7] Otherwise, consider $-\Xi(t)$.

say, where K does not depend on n Our supposition also says there is a positive $m = m(T)$ with $\Xi(t)(\frac{1}{4} + t^2)^{-1} \geqslant m$ for $t \in (2T, 2T + 1)$. Therefore

$$\int_T^\infty \geqslant m \int_{2T}^{2T+1} t^{2n}\, dt > m(2T)^n.$$

Jointing both the above inequalities gives

$$2^{2n} m < K,$$

which is clearly false if we take n large enough. # □

This theorem can be further refined to give a lower bound on the number of zeros on the critical line. The best statement we have is

Theorem 4. [8] *Let $N(T)$ be the number of zeros in the critical strip with $0 < t < T$, and let $N_0(T)$ be the number of zeros on the critical line with $0 < t < T$. Then for large enough T,*

$$N_0(T) \geqslant \alpha N(T),$$

where $\alpha = 0.4077$.

References

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[3] E. Landau. "Über die Hardysche Entdeckung unendlich vieler Nullstellen der Zetafunktion mit reellem Teil $\frac{1}{2}$". German. In: *Mathematische Annalen* 76.2-3 (1915), pp. 212–243. ISSN: 0025-5831. DOI: 10.1007/BF01458139.

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[8] Levinson [4], Conrey [1], et al.