Complex Methods Sheet 1 Question 7

The one with the two branch points

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1 Question

Let $f(z) = (z^2 - 1)^{1/2}$, and consider two different branches of the function $f(z)$:

$f_1(z)$: branch cut $(-\infty, 1] \cup [1, \infty)$, with $f_1(x) = -i\sqrt{1-x^2}$ for real $x \in (-1, 1)$;

$f_2(z)$: branch cut $[-1, 1]$, with $f_2(x) = \sqrt{x^2-1}$ for real $x > 1$.

Find the limiting values of f_1 and f_2 above and below their respective branch cuts. Prove that f_1 is an even function, i.e. $f_1(z) = f_1(-z)$, and that f_2 is odd.

2 Answer

We do everything for f_1, then repeat it for f_2. I do want to stress that, in this topic more than many others, *it is a good idea to work through at least some of the calculations for yourself as well as reading this*, you will understand the processes a lot better!

2.1 f_1

2.1.1 Expression

We need to come up with a way to define f_1 that produces the correct values on the specified interval. Consider the following diagram:

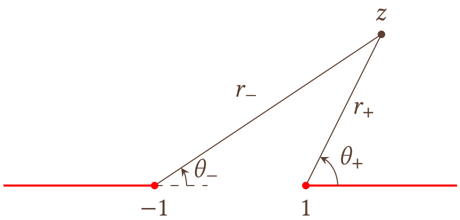

Figure 1: z in general position, defining $r_\pm$ and $\theta_\pm$

We define $r_\pm$ as the distances to z from ± 1, and the angles $\theta_\pm$ are as given in the diagram. If $-1 < x < 1$, therefore, the quantities defined above are given by

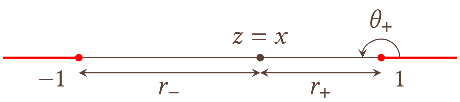

Figure 2: $z = x$ in the interval $(-1, 1)$. θ_- is 0 and not shown

$$r_+ = 1 - x \qquad \theta_+ = \pi$$
$$r_- = 1 + x \qquad \theta_- = 0$$

and then

$$-\sqrt{r_+ r_-}e^{i(\theta_+ + \theta_-)/2} = -\sqrt{(1-x)(1+x)}e^{i(\pi+0)/2} = -i\sqrt{1-x^2}$$

as required. Therefore we take

$$f_1(z) = -\sqrt{r_+ r_-}e^{i(\theta_+ + \theta_-)/2}.$$

2.1.2 Values above and below branch cuts

In theory this is one branch cut, but it is less confusing in this case to consider the halves separately.

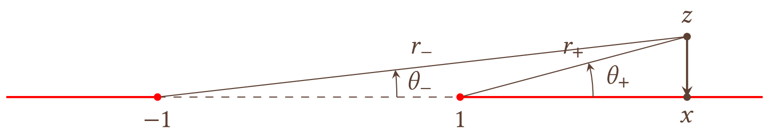

Figure 3: z approaching x on the right part of the branch cut from above

Right branch cut from above Here,

$$r_+ \longrightarrow x-1 \qquad \theta_+ \longrightarrow 0$$
$$r_- \longrightarrow x+1 \qquad \theta_- \longrightarrow 0$$

so as $\varepsilon \downarrow 0$,

$$f_1(x+i\varepsilon) \longrightarrow -\sqrt{(x-1)(x+1)}e^{i(0+0)/2} = -\sqrt{x^2-1}.$$

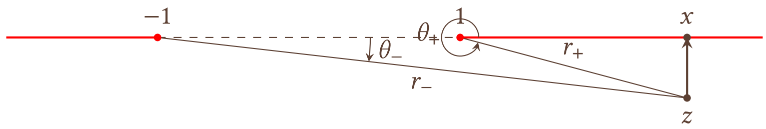

Figure 4: z approaching x on the right part of the branch cut from below

Right branch cut from below Here,

$$r_+ \longrightarrow x-1 \qquad \theta_+ \longrightarrow 2\pi$$
$$r_- \longrightarrow x+1 \qquad \theta_- \longrightarrow 0$$

so as $\varepsilon \downarrow 0$,

$$f_1(x-i\varepsilon) \longrightarrow -\sqrt{(x-1)(x+1)}e^{i(2\pi+0)/2} = +\sqrt{x^2-1}.$$

This makes sense: we expect to have a discontinuity at a branch cut.

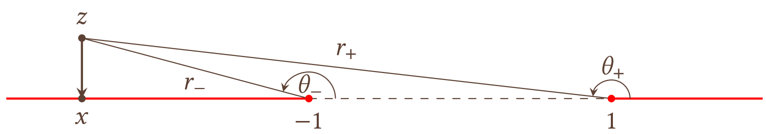

Figure 5: z approaching x on the left part of the branch cut from above

Left branch cut from above

$$r_+ \longrightarrow 1-x \qquad \theta_+ \longrightarrow \pi$$
$$r_- \longrightarrow -1-x \qquad \theta_- \longrightarrow \pi$$

so as $\varepsilon \downarrow 0$,

$$f_1(x+i\varepsilon) \longrightarrow -\sqrt{(1-x)(-1-x)}e^{i(\pi+\pi)/2} = +\sqrt{x^2-1}.$$

This is perhaps not what you might have guessed had you tried to treat the branch cut as a single curve.

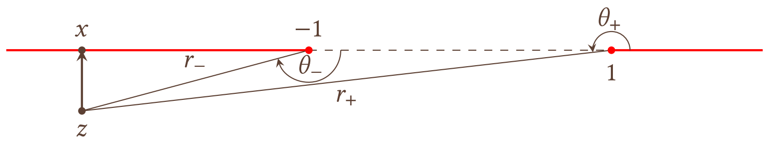

Figure 6: z approaching x on the left part of the branch cut from above

Left branch cut from below

$$r_+ \longrightarrow 1 - x \qquad \theta_+ \longrightarrow \pi$$
$$r_- \longrightarrow -1 - x \qquad \theta_- \longrightarrow -\pi$$

so as $\varepsilon \downarrow 0$,

$$f_1(x - i\varepsilon) \longrightarrow -\sqrt{(1-x)(-1-x)}e^{i(\pi-\pi)/2} = -\sqrt{x^2-1}.$$

2.1.3 Evenness

Here there is only one way to get from z to $-z$: through the middle interval.

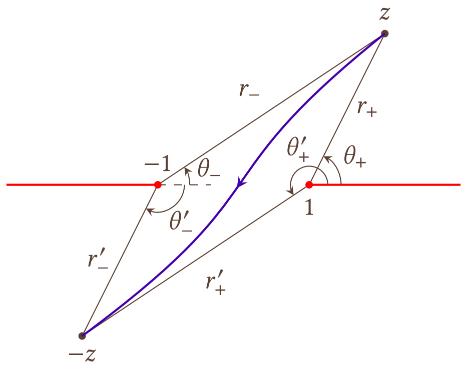

Figure 7: z and $-z$ in general position with $r_\pm, \theta_\pm, r'_\pm, \theta'_\pm$ labelled, the unprimed angles deform continuously into the primed ones along the marked path.

From the diagram, remembering that $\theta_\pm$ should vary continuously as we move along the path,

$$r'_+ = r_- \qquad \theta'_+ = \pi + \theta_-$$
$$r'_- = r_+ \qquad -\theta'_- = \pi - \theta_+$$

If the last result is not clear, remember that θ'_- is positive above the axis too, since it is defined the same way as θ_-; hence we need an extra $-$ on the equality we obtain from geometry.

In particular, this means that

$$r'_+ r'_- = r_+ r_- \qquad \theta'_+ + \theta'_- = \theta_+ + \theta_-,$$

and so

$$f_1(-z) = \sqrt{r'_+ r'_-}e^{i(\theta'_+ + \theta'_-)/2} = \sqrt{r_+ r_-}e^{i(\theta_+ + \theta_-)/2} = f_1(z),$$

and so f_1 is indeed even.

Of course having proven this, the results about the left branch cut follow from those for the right branch cut, but it is good practice to work through both anyway.

2.2 f_2

2.2.1 Expression

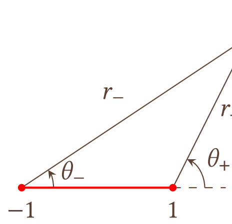

Figure 8: z in general position, defining $r_\pm$ and $\theta_\pm$

We need to define $f_2(z)$ so that $f_2(x) = \sqrt{x^2 - 1}$ for $x > 1$.

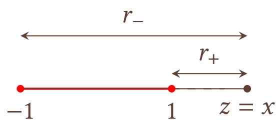

Figure 9: z in the interval $(1, \infty)$ showing $r_\pm, \theta_\pm$: the latter are both 0

$$r_+ = x - 1 \qquad \theta_+ = 0$$
$$r_- = 1 + x \qquad \theta_- = 0$$

and then

$$\sqrt{r_+ r_-} e^{i(\theta_+ + \theta_-)/2} = \sqrt{(x-1)(1+x)} e^{i(0+0)/2} = \sqrt{x^2 - 1}$$

as required. Therefore we take

$$f_2(z) = \sqrt{r_+ r_-} e^{i(\theta_+ + \theta_-)/2}.$$

2.2.2 Values above and below branch cuts

There is definitely only one branch cut here, so we only have to do it once.

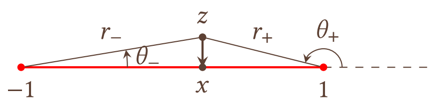

Figure 10: z approaching x on the branch cut from above

Branch cut from above

$$r_+ \longrightarrow 1 - x \qquad \theta_+ \longrightarrow \pi$$
$$r_- \longrightarrow 1 + x \qquad \theta_- \longrightarrow 0$$

so as $\varepsilon \downarrow 0$,

$$f_2(x + i\varepsilon) \longrightarrow \sqrt{(1-x)(1+x)} e^{i(\pi+0)/2} = +i\sqrt{x^2 - 1}.$$

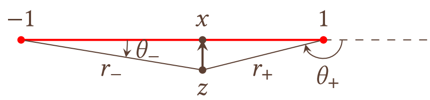

Figure 11: z approaching x on the branch cut from below

Branch cut from below

$$r_+ \longrightarrow 1 - x \qquad \theta_+ \longrightarrow -\pi$$
$$r_- \longrightarrow 1 + x \qquad \theta_- \longrightarrow 0$$

so as $\varepsilon \downarrow 0$,

$$f_2(x + i\varepsilon) \longrightarrow \sqrt{(1-x)(1+x)}e^{i(-\pi+0)/2} = -i\sqrt{x^2-1}.$$

and again we get discontinuity, and this time a preview of the oddness result.

2.2.3 Oddness

We have two ways to get from z to $-z$, we consider the one that passes to the right of 1.[1]

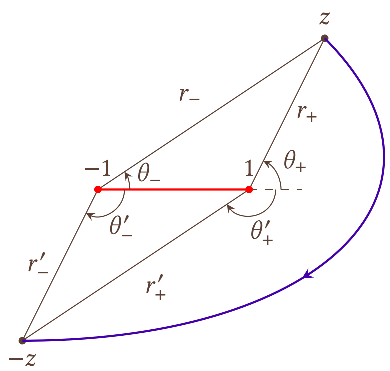

Figure 12: z and $-z$ in general position with $r_\pm, \theta_\pm, r'_\pm, \theta'_\pm$ labelled, the unprimed angles deform continuously into the primed ones along the marked path.

From the diagram,

$$r'_+ = r_- \qquad -\theta'_+ = \pi - \theta_-$$
$$r'_- = r_+ \qquad -\theta'_- = \pi - \theta_+$$

If the $\theta_\pm$ results are not clear, as before remember that $\theta'_\pm$ are positive above the axis too, since they are defined the same way as $\theta_\pm$; hence in both cases this time we need an extra $-$ on the equality we obtain from geometry.

In particular, this means that

$$r'_+ r'_- = r_+ r_- \qquad \theta'_+ + \theta'_- = \theta_+ + \theta_- - 2\pi,$$

and so

$$f_2(-z) = \sqrt{r'_+ r'_-}e^{i(\theta'_+ + \theta'_-)/2} = \sqrt{r_+ r_-}e^{i(\theta_+ + \theta_- - 2\pi)/2} = -f_2(z),$$

and so f_2 is indeed odd.

[1] A worthwhile exercise is to carry out the same calculation for going the other way, and check it agrees.