The Cyclic Product Rule for Partial Derivatives

A geometric interpretation

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Given a function f, all the information given by the first partial derivatives of f is, by linearity of the derivatives, given in the tangent plane of f, that is the plane

$$f(x) + \sum_i h_i \frac{\partial f}{\partial x_i}$$

where h_i are real numbers.

Hence we consider the calculation for a plane, which is easily realised geometrically: see Figure 1. It is intended that the distances be interpreted as signed, in which case the derivation remains true however the plane is oriented, providing that each of a, b, c is nonzero.

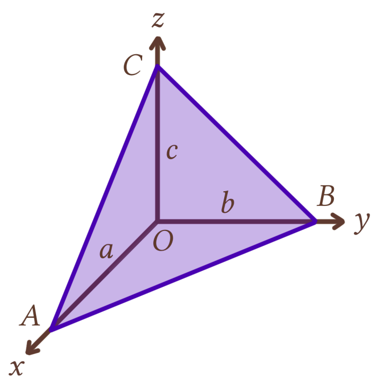

Figure 1: Plane in three dimensions with distances $|OA| = a$ etc. marked. [Picture drawn by JFW]

Then we can read the values of the various partial derivatives off the diagram, as the gradients of the marked lines, that are contained within the coordinate planes:

$$\left.\frac{\partial x}{\partial y}\right|_z = -\frac{a}{b}$$
$$\left.\frac{\partial y}{\partial z}\right|_x = -\frac{b}{c}$$
$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{c}{a}.$$

Multiplying these together gives the cyclic relation

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = \left(-\frac{a}{b}\right)\left(-\frac{b}{c}\right)\left(-\frac{c}{a}\right) = (-1)^3 = -1.$$

Exactly the same proof will work in higher dimensions, but the diagram is rather less accessible!