Proof of Differentiation Under the Integral Sign

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Our main aim in this handout is to prove the following result:

Theorem 1 (Differentiation under the integral sign, variable limits). *Let $f : [a,b] \times (t-\varepsilon, t+\varepsilon) \to \mathbb{R}$ be a continuous function, and that $\partial_2 f(\cdot, t)$ exists and is continuous. Suppose also that $\alpha, \beta \colon (t-\varepsilon, t+\varepsilon) \to [a,b]$ are differentiable with continuous derivatives. For $\tau \in (t-\varepsilon, t+\varepsilon)$, define*

$$I(\tau) = \int_{\alpha(\tau)}^{\beta(\tau)} f(x,\tau)\, dx. \tag{1}$$

Then

$$I'(t) = \beta'(t) f(\beta(t), t) - \alpha'(t) f(\alpha(t), t) + \int_{\alpha(t)}^{\beta(t)} \partial_2 f(x,t)\, dt. \tag{2}$$

Remark 2. Everything we do in the theoretical section of this handout works equally well for one-sided limits and derivatives without any substantial changes; one replaces $(t-\varepsilon, t+\varepsilon)$ by $[0, t+\varepsilon)$, $h \to 0$ by $h \downarrow 0$, and so on.

1 The easy bits

We first divide $I(\tau)$ into a number of integrals for which the derivatives are easier to evaluate:

$$\begin{aligned} I(\tau) &= \int_{\alpha(\tau)}^{\beta(\tau)} f(x,\tau)\, dx \\ &= \int_{\beta(t)}^{\beta(\tau)} f(x,\tau)\, dx - \int_{\alpha(t)}^{\alpha(\tau)} f(x,\tau)\, dx + \int_{\alpha(t)}^{\beta(t)} f(x,\tau)\, dx \\ &= \int_{\beta(t)}^{\beta(\tau)} f(x,t)\, dx - \int_{\alpha(t)}^{\alpha(\tau)} f(x,t)\, dx + \int_{\alpha(t)}^{\beta(t)} f(x,\tau)\, dx + \int_{\beta(t)}^{\beta(\tau)} \big(f(x,\tau) - f(x,t)\big)\, dx - \int_{\alpha(t)}^{\alpha(\tau)} \big(f(x,\tau) - f(x,t)\big)\, dx \\ &=: I_1(\tau) - I_2(\tau) + I_3(\tau) - I_4(\tau) + I_5(\tau). \end{aligned}$$

Now we do the easy cases, which we tackle in two lemmata:

Lemma 3 ($I_1'(t)$ and $I_2'(t)$). *We have*

$$I_1'(t) = \alpha'(t) f(\alpha(t), t), \qquad I_2'(t) = \beta'(t) f(\beta(t), t).$$

Proof. Because α is differentiable and $f(\cdot, t)$ is continuous, this follows immediately from the Fundamental Theorem of Calculus and the chain rule. □

Lemma 4 ($I_4'(t)$ and $I_5'(t)$). *We have $I_4'(t) = I_5'(t) = 0$.*

Proof. We shall consider I_4, since the proof for I_5 is identical in form. Since $I_4(t) = 0$, we need to show that

$$\lim_{h\to 0} \frac{I_4(t+h)}{h} = \lim_{h \to 0} \int_{\alpha(t)}^{\alpha(t+h)} \frac{f(x,t+h) - f(x,t)}{h}\, dx = 0.$$

We have the bound

$$\left| \int_A^B g \right| \leqslant \left| \int_A^B |g| \right| \leqslant |B - A| \sup_{x \text{ between } A \text{ and } B} |g|,$$

which follows straight from the definition of the integral, so we have

$$\int_{\alpha(t)}^{\alpha(t+h)} \frac{f(x,t+h) - f(x,t)}{h}\, dx \leqslant |\alpha(t+h) - \alpha(t)| \sup_{x \text{ between } \alpha(t) \text{ and } \alpha(t+h)} \left| \frac{f(x,t+h) - f(x,t)}{h} \right|.$$

The first term converges to 0 as $h \to 0$ since α is differentiable at t, and the second is bounded as $h \to 0$ since f is continuous and differentiable at t. So $|I_4(t+h)/h| \to 0$ as $h \to 0$, and therefore $I_4'(t) = 0$. □

Hence we are reduced to understanding $I_3(t)$. This is the hard bit.

2 The difficult bit

To complete the proof of Theorem 1, we need to show

Theorem 5 (Differentiation under the integral sign, constant limits). *Let $f\colon [a,b]\times(t-\varepsilon,t+\varepsilon)$ be a continuous function, and that $\partial_2 f(\cdot,t)$ exists and is continuous. Define $F\colon (t-\varepsilon,t+\varepsilon)\to\mathbb{R}$ by*

$$F(\tau)=\int_a^b f(x,\tau)\,dx.$$

Then F is differentiable at t, with

$$F'(t)=\int_a^b \partial_2 f(x,t)\,dx.$$

Notice that the meaning of a and b is slightly different from in the original theorem. This is simply notation, rather than mathematically significant.

However we prove this theorem, we will at some point need the following:

Proposition 6 (Continuity under the integral sign). *Let $g\colon [a,b]\times(t-\varepsilon,t+\varepsilon)\to\mathbb{R}$ be continuous. Then $G(\tau):=\int_a^b g(x,\tau)\,dx$ is continuous at t.*

Proof. Alas, we shall proceed by contradiction. We wish to show that

$$\lim_{h\to 0}\int_a^b (g(x,t+h)-g(x,t))\,dx=0$$

We write this integral as $J(h)$ for brevity. Supposing instead that $J(h)\nrightarrow 0$, there is a "bad η" and a sequence $(h_n)_{n=0}^{\infty}$ so that $h_n\to 0$ and $|J(h_n)|>\eta$ for every n.

For each n, there must be a point $x_n\in(a,b)$ so that $|g(x_n,t+h_n)-g(x_n,t)|>\eta/(b-a)$ (or by the trivial bound $\left|\int_a^b f\right|\leqslant (b-a)\sup_{[a,b]}|f|$, we would have $|J(h_n)|\leqslant\eta$).

By Bolzano–Weierstrass, there is a point $X\in[a,b]$ and a subsequence $(x_{n(k)})_{k=0}^{\infty}$ so that $x_{n(k)}\to X$ as $k\to\infty$. Since g is continuous, we have $g(x_{n(k)},t+h_{n(k)})\to g(X,t)$ and $g(x_{n(k)},t)\to g(X,t)$ as $k\to\infty$. For all k,

$$\frac{\eta}{b-a}<|g(x_{n(k)},t+h_{n(k)})-g(x_{n(k)},t)|\leqslant|g(x_{n(k)},t+h_{n(k)})-g(X,t)|+|g(X,t)-g(x_{n(k)},t)|.$$

But we just noted that the two terms on the right converge to 0 as $k\to\infty$, which gives a contradiction. Therefore we must have $J(h)\to 0$, and the result follows. □

This proof is quite similar to the Analysis I proof that continuous functions are integrable. Both are more naturally done using uniform continuity, covered in Analysis II.[1] We required the interval to be compact to use Bolzano–Weierstrass.

This type of proof is surprisingly difficult to nail down: it took the author several attempts to produce a legitimate version. Having done so, however, it makes the theorem quite simple to prove:

Proof of Theorem. By definition,

$$F'(t)=\lim_{h\to 0}\frac{1}{h}\left(\int_a^b f(x,t+h)\,dx-\int_a^b f(x,t)\,dx\right)=\lim_{h\to 0}\int_a^b\frac{f(x,t+h)-f(x,t)}{h}\,dx,$$

and we would like to show that this is $\int_a^b\partial_2 f(x,t)\,dx$.

Since f is continuous in both arguments and differentiable at t in the second, the function $g\colon[a,b]\times(-\varepsilon,\varepsilon)\to\mathbb{R}$ given by

$$g(x,h)=\begin{cases}(f(x,\tau)-f(x,t))/h & h\neq 0\\ \partial_2 f(x,t) & h=0\end{cases}$$

is continuous. We can now apply Proposition 6 to conclude that $\lim_{h\to 0}\int_a^b(f(x,t+h)-f(x,t))/h\,dx=\int_a^b\partial_2 f(x,t)\,dx$, as required. □

It follows immediately that

$$I_3'(t)=\int_{\alpha(t)}^{\beta(t)}\partial_2 f(x,t)\,dx.$$

[1] West Bromwich Albion: Nil.[2]

[2] No, I have no idea what this one means either. Something to do with foot-ball?

3 Advertisement: More general results

The best possible result for the (proper) Riemann integral is probably the following:

Theorem 7 (Differentiation under the integral sign for integrable functions). *Let $f\colon [a,b]\times(t-\varepsilon,t+\varepsilon)\to\mathbb{R}$, and suppose that*

- $f(\cdot,\tau)$ *is integrable for each* $\tau\in(t-\varepsilon,t+\varepsilon)$,
- $\partial_2 f(x,t)$ *exists for each* $x\in[a,b]$ *and* $\partial_2 f(\cdot,t)$ *is integrable,*
- *There is* K *so that* $\left|(f(x,\tau)-f(x,t))/(\tau-t)\right|\leqslant K$ *for* $0<|\tau-t|<\varepsilon$.

Then

$$F(\tau) := \int_a^b f(x,\tau)\,dx$$

has derivative

$$F'(t) = \int_a^b \partial_2 f(x,t)\,dx.$$

These are in some sense the weakest possible conditions we could ask for on f for the derivatives on both sides of to make sense. The proof works in a similar way to Theorem 5, but showing that the integral is continuous is rather more difficult, so we shall not do so here.

A far more powerful result is available for the Lebesgue integral:

Theorem 8 (Differentiation under the integral sign for the Lebesgue integral). *Let E be a measurable subset*[3] *of $\mathbb{R}$ and $f\colon E\times(t-\varepsilon,t+\varepsilon)\to\mathbb{R}$. Suppose that*

- $f(\cdot,\tau)$ *is Lebesgue-integrable on E for each* $\tau\in(t-\varepsilon,t+\varepsilon)$,
- $\partial_2 f(x,t)$ *exists for each* $x\in E$,
- $\left|(f(x,\tau)-f(x,t))/(\tau-t)\right|\leqslant g$ *is uniformly bounded by some Lebesgue-integrable g with* $\int_E g<\infty$ *for* $0<|\tau-t|<\varepsilon$.

Then

$$F(\tau) := \int_E f(x,\tau)\,dx$$

has derivative

$$F'(t) = \int_E \partial_2 f(x,t)\,dx.$$

Notice that we do not have to assume that $\partial_2 f(\cdot,t)$ is Lebesgue-integrable: that comes for free in Lebesgue's theory!

Remark 9. To include variable limits in either of these, one has to also ask that f is continuous in its first argument at $\alpha(t)$ and $\beta(t)$ (one still has to use some version of the Fundamental Theorem of Calculus (DI)), which makes for slightly messier hypotheses.

4 $\int_0^\infty \frac{\sin x}{x}\,dx$

One of the most common applications of differentiation under the integral sign is to prove that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

by considering the function

$$I(\lambda,\alpha) := \int_0^\infty e^{-\alpha x}\sin\lambda x/x\,dx.$$

However, none of the results we have discussed so far are good enough for $I(\lambda,\alpha)$, which is not Lebesgue-integrable when $\alpha=0$. It must instead be treated by ad-hoc methods. We suppose that $\lambda>0$, since if $\lambda<0$ we can just use $\sin(-y)=-\sin y$.

Our result *does* apply to

$$I(\lambda,\alpha,X) := \int_0^X e^{-\alpha x}\frac{\sin\lambda x}{x}\,dx.$$

[3] Suffice to say for the time being that these are the "right" subsets of $\mathbb{R}$ on which to integrate. Examples include $\mathbb{R}$, any interval, and any countable union of intervals, which is plenty to be getting on with.

We obtain

$$\frac{\partial}{\partial\lambda}\int_0^X e^{-\alpha x}\frac{\sin\lambda x}{x}\,dx = \int_0^X e^{-\alpha x}\cos\lambda x\,dx = \cdots = \frac{\alpha}{\lambda^2+\alpha^2} + e^{-\alpha X}\frac{-\alpha\cos\lambda X + \lambda\sin\lambda X}{\lambda^2+\alpha^2}.$$

Also, the integrand is continuous for $\lambda \in [0,\infty)$ and $I(0,\alpha,X) = 0$, so integrating,

$$I(\lambda,\alpha,X) = \frac{\pi}{2} - \arctan\left(\frac{\alpha}{\lambda}\right) + e^{-\alpha X}\int_0^\lambda \frac{-\alpha\cos yX + y\sin yX}{y^2+\alpha^2}\,dy.$$

This concludes the differentiation part of the argument; we have completely avoided the improper integral so far. To return to $I(\lambda,\alpha)$, we take $X\to\infty$. The second term is bounded above by $e^{-\alpha X}\big(\log\sqrt{1+(\lambda/\alpha)^2} + \arctan(\lambda/\alpha)\big) \to 0$, so we have

$$I(\lambda,\alpha) = \int_0^\infty e^{-\alpha x}\frac{\sin\lambda x}{x}\,dx = \frac{\pi}{2} - \arctan\left(\frac{\alpha}{\lambda}\right).$$

It remains to show that $I(\lambda,0) = \int_0^\infty \sin\lambda x/x\,dx$ exists, and that $I(\lambda,\alpha)\to I(\lambda,0)$ as $\alpha\to 0$. A change of variables shows that the value of λ is no longer relevant, in that $I(\lambda,\alpha) = I(1,\alpha/\lambda)$, so we now take $\lambda = 1$.

Firstly, putting $\alpha = 0$ gives[4]

$$I(1,0) = \int_0^\infty \frac{1-\cos x}{x^2}\,dx,$$

which is nonnegative since the integrand is. Since $1-\cos x < x^2/2$ and $1-\cos x < 2$, we can split the integral at 1, and find that

$$I(1,0) = \int_0^1 \frac{x^2}{2x^2}\,dx + \int_1^\infty \frac{dx}{x^2},$$

both of which converge, so $I(1,0)$ converges.

It remains to show that $I(1,\alpha)\to I(1,0)$. Integrating by parts, we have

$$\begin{aligned} I(1,\alpha,X) &= \left[\frac{1-e^{-\alpha x}}{x}(1-\cos x)\right]_0^X + \int_0^X \left(\frac{1-e^{-\alpha x}}{x^2} - \frac{\alpha e^{-\alpha x}}{x}\right)(1-\cos x)\,dx \\ &= \frac{1-e^{-\alpha X}}{X}(1-\cos X) + \int_0^X (1-e^{-\alpha x})\frac{1-\cos x}{x^2}\,dx - \alpha\int_0^X e^{-\alpha x}\frac{1-\cos x}{x}\,dx, \end{aligned}$$

and taking the limit as $X\to\infty$,

$$I(\lambda,\alpha,X) = \int_0^\infty (1-e^{-\alpha x})\frac{1-\cos x}{x^2}\,dx - \alpha\int_0^\infty e^{-\alpha x}\frac{1-\cos x}{x}\,dx.$$

Both of these integrals are positive. We now find upper bounds to which we can apply the comparison test.

We can divide the first integral at α^k, whence we obtain

$$\int_0^{\alpha^k} (1-e^{-\alpha x})\frac{1-\cos x}{x^2}\,dx + \int_{\alpha^k}^\infty (1-e^{-\alpha x})\frac{1-\cos x}{x^2}\,dx.$$

The first of these is bounded above by

$$\alpha\int_0^{\alpha^k} \frac{1-\cos x}{x}\,dx < \alpha\alpha^k M,$$

since $(1-\cos x)/x$ is bounded above by M, say. This converges to 0 as $\alpha\to 0$ provided that $k+1>0$. In the second, the numerator is bounded above by 1, so the integral is bounded above by

$$\int_{\alpha^k}^\infty \frac{dx}{x^2} = \alpha^{-k},$$

which converges to 0 as $\alpha\to 0$ provided that $k<0$.

The final integral can be dealt with using an exceptionally crude bound: one can check that for $x>0$, $1-\cos x < 2\sqrt{x}$. Hence

$$\alpha\int_0^\infty e^{-\alpha x}\frac{1-\cos x}{x}\,dx < 2\alpha\int_0^\infty x^{-1/2}e^{-\alpha x}\,dx = 2\alpha^{1/2}\int_0^\infty u^{-1/2}e^{-u}\,du,$$

putting $u = x/\alpha$. It is easy to check by examining $u\to 0$ and $u\to\infty$ that the latter integral converges, so this also converges to 0 as $x\to 0$.

Hence we have shown that $I(1,\alpha)\to I(1,0)$ as $\alpha\to 0$, as required. It follows that

$$I(1,0) = \lim_{\alpha\to 0} I(1,\alpha) = \lim_{\alpha\to 0}\frac{\pi}{2} - \arctan\alpha = \frac{\pi}{2}.$$

[4] Those suspicious that something has been swept under the rug here are encouraged to carry out the integration by parts for $\alpha = 0$ themselves.