Euler's Trigonometrical Products and Partial Fraction Formulae

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§ 1 Throughout, z is a complex number. Any definition of the trigonometrical functions that works for complex numbers may be used, but for definiteness you may assume the definition is via the Maclaurin series. Our aim in this handout is to derive expansions for the trigonometrical (and hyperbolic) functions similar to those that polynomials and rational functions have: as a product over their roots in the case of the continuous functions sine and cosine, or for the others, a "partial fractions" expansion over their singularities. Achieving this is rather more difficult than for polynomials: since all six trigonometrical functions are periodic, they all have infinitely many zeros or infinitely many singularities.

§ 2 We start by proving the formula

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right), \tag{1}$$

famously first derived by Euler,[1] and first properly justified by Weierstrass.[2]

There is a straightforward argument that owes much to a well-known proof of the Wallis product,

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{2k}{2k-1} \frac{2k}{2k+1}, \tag{2}$$

using integrals.[4,5] We begin with

$$I_n(z) \coloneqq \int_0^{\pi/2} \cos 2zt \cos^n t \, dt.$$

Since $I_0(z) = \sin \pi z/(2z)$ and $I_0(0) = \pi/2$, we have

$$\frac{I_0(z)}{I_0(0)} = \frac{\sin \pi z}{\pi z}.$$

We now derive a recurrence relation for $I_n(z)$. This goes in the same way as usual: we integrate by parts.

$$\begin{aligned} I_n(z) &= \int_0^{\pi/2} \cos 2zt \cos^n t \, dt \\ &= \left[\frac{1}{2z} \sin 2zt \cos^n t\right]_0^{\pi/2} + \frac{n}{2z} \int_0^{\pi/2} \sin 2zt \sin t \cos^{n-1} t \, dt \\ &= 0 + \left[-\frac{n}{4z^2} \cos 2xt \sin t \cos^{n-1} t\right]_0^{\pi/2} \\ &\quad - \frac{n}{4z^2} \int_0^{\pi/2} \cos 2zt \Big(\cos^n t - (n-1) \sin^2 t \cos^{n-2} t\Big) \, dt \\ &= 0 + \frac{n^2}{4z^2} I_n(z) - \frac{n(n-1)}{4z^2} I_{n-2}(x) \end{aligned}$$

since $\sin^2 t + \cos^2 t = 1$. Rearranging gives

$$I_{n-2}(z) = \frac{n^2 - 4z^2}{n(n-1)} I_n(x).$$

Naturally the same is true for $z = 0$, although the derivation will need to be a little different. Dividing one relation by the other then gives

$$\frac{I_{n-2}(z)}{I_{n-2}(0)} = \frac{n^2 - 4z^2}{n^2} \frac{I_n(z)}{I_n(0)} = \left(1 - \frac{4z^2}{n^2}\right) \frac{I_n(z)}{I_n(0)}.$$

Iterating this $m - 1$ times from $n = 2$ gives

$$\frac{I_0(z)}{I_0(0)} = \frac{I_{2m}(z)}{I_{2m}(0)} \prod_{k=1}^{m} \left(1 - \frac{4z^2}{(2k)^2}\right) = \frac{I_{2m}(z)}{I_{2m}(0)} \prod_{k=1}^{m} \left(1 - \frac{z^2}{k^2}\right).$$

Finally, we need to show that $I_{2m}(z)/I_{2m}(0) \to 1$ as $m \to \infty$.

It is possible to derive this without recourse to general results, but we shall instead take the opportunity to prove something genuinely useful:[6]

Theorem 1 (Approximation to the identity). *Let $f : (a, b) \to \mathbb{C}$ be continuous and bounded, and let $\phi_n : (a, b) \to \mathbb{R}$ be a sequence of continuous functions that satisfy the following:*

1. *$\phi_n(x) \geqslant 0$ for all $x \in (a, b)$,*
2. *$\int_a^b \phi_n = 1$,*
3. *There is $c \in [a, b]$ so that given any $\varepsilon > 0$ and $\delta > 0$, there is N so that $\int_a^{c-\delta} \phi_n + \int_{c+\delta}^b \phi_n < \varepsilon$ for any $n > N$.*

[1] There are various derivations of this in Euler's vast œuvre; the first proof is in the celebrated paper [2] (in which Euler shows that $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$), while one of the clearest is in his (still very readable) textbook [3, Cap. XI, esp. § 158].

[2] His general theory of expanding a complex-analytic function[3] as an infinite product is enunciated in [13].

[3] Precisely what these are and why they are the best thing ever will be explained in IB Complex Analysis; alas, we have no time for such joys at this juncture.

[4] We are aware of at least two expositions of this type of proof [1, 11], and while both use similar ideas to the following, the proof of convergence we shall employ instead uses a very useful theorem from analysis, rather than results about specific functions.

[5] Many other proofs exist: [5, 6, 7, 8, 9] are a selection of recent elementary ones, with varying levels of sophistication.

[6] And which, indeed, we will use again for other integrals in the sequel.

Then

$$\lim_{n\to\infty}\int_a^b f\phi_n = f(c).$$

Heuristically, the third criterion forces the ϕ_n to bunch up around $x=c$; combining this with the first two conditions forces the same area to become more and more concentrated around $x=c$, so the values of f around c are a larger and larger proportion of the value of the integral. Since f is continuous, for small enough δ these values are very close to $f(c)$. The proof will essentially be a formalisation of this argument.

Why (apart from sheer delight in generality) would we prove this over a seemingly simpler result? Firstly, as occurs frequently in mathematics, the general result is in some sense easier to prove than special cases, because one pares down the required properties to the essentials, and will no longer be distracted by irrelevant algebraic properties of the objects involved. Secondly, having proved this once, we can apply it to lots of different functions, rather than constantly working *ad hoc*.

Proof. Pick $\varepsilon > 0$. Using the second condition, we have

$$\begin{aligned}\left|\int_a^b f\phi_n - f(0)\right| &= \left|\int_a^b (f(x)-f(0))\phi_n(x)\,dx\right| \\ &\leqslant \left|\int_{c-\delta}^{c+\delta} (f(x)-f(0))\phi_n(x)\,dx\right| \\ &\quad + \left|\int_a^{c-\delta} (f(x)-f(0))\phi_n(x)\,dx\right| \\ &\quad + \left|\int_{c+\delta}^{b} (f(x)-f(0))\phi_n(x)\,dx\right|\end{aligned}$$

Since f is continuous, we can choose δ small enough that $|f(x)-f(0)| < \varepsilon/2$ for every $x \in (c-\delta, c+\delta)$. Then using the triangle inequality, $\left|\int g\right| \leqslant \int |g|$, and the first condition in the theorem,

$$\begin{aligned}\left|\int_{c-\delta}^{c+\delta} (f(x)-f(0))\phi_n(x)\,dx\right| &\leqslant \int_{c-\delta}^{c+\delta} |f(x)-f(0)|\phi_n(x)\,dx \\ &\leqslant \frac{\varepsilon}{2}\int_{c-\delta}^{c+\delta}\phi_n(x)\,dx \\ &\leqslant \frac{\varepsilon}{2}\int_a^b \phi_n(x)\,dx = \frac{\varepsilon}{2}\end{aligned}$$

Similarly, f is bounded, by M say, so $|f(x)-f(0)| \leqslant 2M$, and the latter two integrals can also be bounded by using the triangle inequality:

$$\begin{aligned}\left|\int_a^{c-\delta} (f(x)-f(0))\phi_n(x)\,dx\right| &\leqslant \int_a^{c-\delta} |f(x)-f(0)|\phi_n(x)\,dx \\ &\leqslant 2M\int_a^{c-\delta}\phi_n \\ \left|\int_{c+\delta}^{b} (f(x)-f(0))\phi_n(x)\,dx\right| &\leqslant \int_{c+\delta}^{b} |f(x)-f(0)|\phi_n(x)\,dx \\ &\leqslant 2M\int_{c+\delta}^{b}\phi_n,\end{aligned}$$

and by the third condition in the theorem, we can choose N large enough that $\int_a^{c-\delta}\phi_n + \int_{c+\delta}^b \phi_n \leqslant \varepsilon/(4M)$ for $n > N$. Thus

$$\left|\int_a^b f\phi_n - f(0)\right| < \frac{\varepsilon}{2} + 2M\frac{\varepsilon}{4M} = \varepsilon$$

for $n > N$. Since this holds for any $\varepsilon > 0$, the result follows. □

In particular, since $\phi_n(t) = (\cos^n t)/I_n(0)$ satisfies the conditions of the theorem on $(0,\pi/2)$ and $f(t) = \cos 2zt$ is continuous and bounded with $f(0) = 1$, the required limit is 1. The Euler product follows.

§ 3 In exactly the same way, we can show using the odd-m integrals that

$$\cos \pi z = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{(k-1/2)^2}\right). \tag{3}$$

§ 4 We may obtain other partial fractions expansions from the product. Taking the logarithmic derivative of the finite product,[7]

$$\pi\cot\pi z - \frac{1}{z} = \frac{I_0'(z)}{I_0(z)} = \sum_{k=1}^{m}\left(\frac{1}{z-k} + \frac{1}{z+k}\right) + \frac{I_{2m}'(z)}{I_{2m}(z)}.$$

The last term can be written as

$$\frac{I_{2m}'(z)}{I_{2m}(0)}\frac{I_{2m}(0)}{I_{2m}(z)},$$

so once again we can use the theorem, with $\phi_n(t) = (\cos^n t)/I_n(0)$ and $f(t) = -2t\sin 2zt$, from which we immediately see that the limit of the first fraction is 0 as $m \to \infty$. The second fraction converges to 1 as before, whence

$$\pi\cot\pi z = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{z-k} + \frac{1}{z+k}\right), \tag{4}$$

a formula also known to Euler.[8,9] A further differentiation gives

$$\pi^2\csc^2\pi z = \sum_{k=-m}^{m}\frac{1}{(z-k)^2} - \frac{I_{2m}''(z)}{I_{2m}(z)} + \left(\frac{I_{2m}'(z)}{I_{2m}(z)}\right)^2,$$

[7] Those alarmed by taking the logarithm of a product of possibly complex numbers will lose nothing by simply beginning with the expression f'/f and cancelling.

[8] First published in [4] (special case of a formula in § 9, explicitly given in § 17), and also found in [3, Cap. X, esp. § 178]

[9] See also my IB Methods handout *Fourier Series/The Sine Product Formula/A Cotangent Series* for another derivation of this result using Fourier series.

and exactly the same argument for $f(t) = -(2t)^2 \sin 2zt$ shows that both remainder terms converge to 0, whence

$$\pi^2 \csc^2 \pi z = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}. \tag{5}$$

§ 5 Similar results are obtainable from the cosine product by the same process, namely

$$\pi \tan \pi z = -\sum_{k=1}^{\infty} \left(\frac{1}{z-(k-1/2)} + \frac{1}{z+(k-1/2)} \right) \tag{6}$$

and

$$\pi^2 \sec^2 \pi z = \sum_{k=-\infty}^{\infty} \frac{1}{(z-(k-1/2))^2}. \tag{7}$$

§ 6 A natural question is "what about the cosecant and the secant?". We can obtain both from trigonometrical identities involving functions we already know about. In particular,

$$\csc \theta = \tfrac{1}{2} \cot \tfrac{1}{2}\theta - \tfrac{1}{2} \cot \tfrac{1}{2}(\theta + \pi),$$

which could have been guessed from the graph. It is straightforward to check that it really is true:[10]

$$\begin{aligned} \cot \tfrac{1}{2}\theta - \cot \tfrac{1}{2}(\theta+\pi) &= \frac{\cos(\theta/2)}{\sin(\theta/2)} - \frac{\cos(\theta/2+\pi/2)}{\sin(\theta/2+\pi/2)} \\ &= \frac{\cos(\theta/2)}{\sin(\theta/2)} + \frac{\sin(\theta/2)}{\cos(\theta/2)} \\ &= \frac{\cos^2(\theta/2)+\sin^2(\theta/2)}{\sin(\theta/2)\cos(\theta/2)} \\ &= \frac{2}{\csc\theta}. \end{aligned}$$

Since $\sec \theta = \csc(\pi - \theta)$, we also obtain

$$\sec\theta = \tfrac{1}{2}\cot\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi\right) - \tfrac{1}{2}\cot\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\pi\right),$$

whence some calculation gives

$$\pi \csc \pi z = \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{z-k} + \frac{1}{z+k} \right) \tag{8}$$

$$\pi \sec \pi z = \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{z-(k-1/2)} - \frac{1}{z+(k-1/2)} \right). \tag{9}$$

(The secant is quite frequently the odd one out in such formulae: it is also the only one whose Taylor expansion does involve the Bernoulli numbers.[11])

§ 7 Finally, we can give versions of all of these formulae to the hyperbolic functions. Since we have allowed z to be complex throughout, we can get these almost for free:

$$\sin iz = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = -i\frac{e^{-z}-e^{z}}{2} = i \sinh z$$

and similarly $\cosh z = \cos iz$, so we obtain immediately the product formulae

$$\sinh \pi z = \pi z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2}\right) \tag{10}$$

$$\cosh \pi z = \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{(k-1/2)^2}\right) \tag{11}$$

and the partial fractions expansions

$$\pi \coth \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z-ik} + \frac{1}{z+ik} \right) \tag{12}$$

$$= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2+k^2} \tag{13}$$

$$\pi^2 \operatorname{csch}^2 \pi z = \sum_{k=-\infty}^{\infty} \frac{1}{(z-ik)^2} \tag{14}$$

$$\pi \tanh \pi z = \sum_{k=1}^{\infty} \left(\frac{1}{z-i(k-1/2)} + \frac{1}{z+i(k-1/2)} \right) \tag{15}$$

$$= \sum_{k=1}^{\infty} \frac{2z}{z^2+(k-1/2)^2} \tag{16}$$

$$\pi^2 \operatorname{sech}^2 \pi z = \sum_{k=-\infty}^{\infty} \frac{1}{(z-i(k-1/2))^2} \tag{17}$$

$$\pi \operatorname{csch} \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{z-ik} + \frac{1}{z+ik} \right) \tag{18}$$

$$= \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k \frac{2z}{z^2+k^2} \tag{19}$$

$$\pi \operatorname{sech} \pi z = \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{z-i(k-1/2)} - \frac{1}{z+i(k-1/2)} \right) \tag{20}$$

$$= \sum_{k=1}^{\infty} (-1)^k \frac{2k-1}{z^2+(k-1/2)^2} \tag{21}$$

§ 8 Finally, we make a brief remark about convergence. The reader might be puzzled that we have written some of the sums as doubly infinite and others as infinite in one direction. This stems from the definition of a doubly infinite sum being

$$\sum_{k=-\infty}^{\infty} f(k) := \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{k=-m}^{n} f(k),$$

implicitly requiring that the double limit has a value independent of the ways in which $m, n \to \infty$. If the summand does not decay fast enough, this may not happen. On the other hand, if $\sum_{k=-\infty}^{\infty} |f(k)|$ is finite, the doubly infinite sum will always make sense;[12] such a sum is called *absolutely convergent*. In particular, $1/(z-k) = -1/k + O(z/k^2)$ as $k \to \pm\infty$, which does not decay fast enough to be absolutely

[10] The reader may like to consider whether logarithmic differentiation of $\cot \frac{1}{2}\theta = \cos \frac{1}{2}\theta / \sin \frac{1}{2}\theta$ gives a simpler derivation.
[11] See my IB Methods handout *The Riemann Zeta Function at Positive Even Integers* for details.
[12] This should be proven in Analysis I.

convergent, so while the single limit $\lim_{n\to\infty}\sum_{k=-m}^{m}\frac{1}{z-k}$ makes sense, $\sum_{k=-\infty}^{\infty}\frac{1}{z-k}$ does not.

To resolve this issue, it is sometimes possible to massage the series to make it converge absolutely. In this case, we note that

$$\frac{1}{z-k}+\frac{1}{z+k}=\frac{1}{z-k}+\frac{1}{k}+\frac{1}{z+k}-\frac{1}{k},$$

and now $1/(z-k)+1/k=z/k^2+o(z/k^2)$ for large k, and $1/k^2$ is absolutely convergent (provided we leave out the term with $k=0$). It is now possible to write the series as a well-defined doubly-infinite sum, with one singularity in each summand:

$$\pi\cot\pi z=\frac{1}{z}+\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(\frac{1}{z-k}+\frac{1}{k}\right). \tag{22}$$

Such a "partial fractions" expansion of an analytic function is called a *Mittag-Leffler expansion.*[13]

Analogous considerations apply to the products, although the situation is more complicated. Weierstrass's results give

$$\sin\pi z=\pi z\prod_{k\in\mathbb{Z}\setminus\{0\}}\left(1-\frac{z}{k}\right)e^{z/k}; \tag{23}$$

the extra $e^{z/k}$ are called *elementary factors*, and improve the convergence properties of the product in much the same way as the extra $1/k$ did in the cotangent series.

References

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[13] After Mittag-Leffler's results on the existance of such expansions in a series of papers, beginning with the basic result in [10]. The reader is referred to [12] for details of the subsequent development. Note that Mittag-Leffler is one person, as is Levi-Civita (and Ricci-Curbastro, but objects named after him are normally just called Ricci somethingorothers).