Series Solutions: The Method of Frobenius

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Suppose that we have a second-order differential equation,

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0. \tag{1}$$

We are interested in the behaviour of this solution near a particular point a, and then in constructing series solutions centred at a. Therefore the first thing to do is consider the behaviour of the differential equation, and in particular the coefficients, near $x = a$.

Notice that the leading coefficient (i.e. that of y'') is always 1. This is essential for the method to work properly.

1 Classification of Singularities

We always suppose that p and q can be expanded about a in a series of integer powers of $(x - a)$.

We say that a function f is *regular* at a if $\lim_{x\to a} f(x)$ exists and is finite. A function that is not regular is called *singular*.

Any continuous function is clearly regular. $\sin x/x$ is also regular at $x = 0$, but $\sin x/x^2$ is singular at $x = 0$.

- If p and q are regular at a, a is called an *ordinary point*.
- If p or q is singular at a, but $(x - a)p(x)$ and $(x - a)^2 q(x)$ are regular, a is called a *regular singular point*.
- If at least one of $(x - a)p(x)$ or $(x - a)^2 q(x)$ is singular at $x = a$, then a is called an *irregular singular point*, and we say no more about them here.[1]

The reason for these distinctions will become evident when we do the actual calculations.[2]

For ease of algebra, from now on we assume that $a = 0$. If $a \neq 0$, one replaces all instances of x in the following by $x - a$.

2 Ordinary Points

At an ordinary point, we have Taylor series expansions for p and q:

$$p(x) = \sum_{k=0}^{\infty} p_k x^k, \qquad q(x) = \sum_{k=0}^{\infty} q_k x^k. \tag{2}$$

We now show that we can produce a solution to the equation in the form of a Taylor series for y: suppose that

$$y(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{3}$$

Then

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k \tag{4}$$

$$y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = \sum_{k=0}^{\infty} (k+1)(k+2) a_{k+2} x^k, \tag{5}$$

where we have shifted the indices to make the sums all start at $k = 0$, which simplifies what we do next. We may substitute these into the differential equation:

$$0 = \sum_{k=0}^{\infty} (k+1)(k+2) a_{k+2} x^k + p(x) \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k + q(x) \sum_{k=0}^{\infty} a_k x^k. \tag{6}$$

Inserting the series expansions of p and q gives

$$0 = \sum_{k=0}^{\infty} (k+1)(k+2) a_{k+2} x^k + \left(\sum_{n=0}^{\infty} p_n x^n \right) \left(\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m \right) + \left(\sum_{n=0}^{\infty} q_n x^n \right) \left(\sum_{m=0}^{\infty} a_m x^m \right). \tag{7}$$

To multiply two power series, we use the *Cauchy product formula*,

$$\left(\sum_{i=0}^{\infty} A_i x^i \right) \left(\sum_{j=0}^{\infty} B_j x^j \right) = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} A_i B_j \right) x^k = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} A_i B_{k-i} \right) x^k. \tag{8}$$

[1] Solutions in the vicinity of irregular singular points may in many cases be constructed by the *Liouville–Green method*, which will be covered in Part II's Asymptotic Methods.

[2] The nature of the point ∞ may be found by substituting $X = 1/x$: one can show that Möbius transformations do not change whether points are regular or singular.

This may be proven, at least formally, by writing out the terms in an array with coordinates (i,j) and noting that the coefficients of x^k are given by the diagonal with $i+j=k$. Moreover, if the original series converge, so does the product series, so this is in a sense as nice as we could hope for.

Applying this, we obtain

$$0=\sum_{k=0}^{\infty}\left((k+1)(k+2)a_{k+2}+\sum_{m=0}^{k}\big(p_{k-m}(m+1)a_{m+1}+q_{k-m}a_m\big)\right)x^k, \tag{9}$$

where all the sums are the same, so we have grouped all the terms together. Implausible though it may seem, we are almost there: since this equation is supposed to hold for x in an extended region, we can equate coefficients of x^k, just as we do for polynomials. This gives us an infinite set of equations,

$$0=(k+1)(k+2)a_{k+2}+\sum_{m=0}^{k}\big(p_{k-m}(m+1)a_{m+1}+q_{k-m}a_m\big). \tag{10}$$

Has this actually helped? In fact a minor miracle has occured: each equation expresses a_{k+2} in terms of a_m for m at most $k+1$, so this is a recurrence relation that may be solved for a_{k+2} provided we have a certain number of a_k specified already. Writing out the first few,

$$0=2a_2+p_0a_1+q_0a_0 \tag{11}$$
$$0=6a_3+2p_0a_2+(p_1+q_0)a_1+q_1a_0 \tag{12}$$
$$0=12a_4+3p_0a_3+(2p_1+q_0)a_2+(p_2+q_1)a_1+q_2a_0 \tag{13}$$
$$\vdots$$

we see that specifying the values of a_0 and a_1, i.e. $y(0)$ and $y'(0)$, is sufficient. Moreover, both can be specified independently, which clearly gives two linearly independent solutions. Hence we conclude

Result 1. *If $x=0$ is an ordinary point, with p and q having series expansions given by* (2)*, then* (1) *has two linearly independent power series solutions $\sum_{k=0}^{\infty}a_kx^k$, where the a_k satisfy* (9).

3 Regular Singular Points

Suppose now that we instead have a regular singular point at 0, so

$$p(x)=\frac{P}{x}+\sum_{k=0}^{\infty}p_kx^k, \qquad q(x)=\frac{Q}{x^2}+\frac{Q'}{x}+\sum_{k=0}^{\infty}q_kx^k. \tag{14}$$

Can we have a power series solution to (1) now? Let's substitute the first few terms $a_0+a_1x+a_2x^2+O(x^3)$ in and see what happens:

$$0=(2a_2+O(x))+\left(\frac{P}{x}+p_0+O(x)\right)(a_1+2a_2x+O(x^2))+\left(\frac{Q}{x^2}+\frac{Q'}{x}+q_0+O(x)\right)(a_0+a_1x+a_2x^2+O(x^3)) \tag{15}$$
$$=\frac{Qa_0}{x^2}+\frac{(P+Q)a_1+a_0}{x}+(2P+Q+2)a_2+(Q'+p_0)a_1+q_0a_0+O(x). \tag{16}$$

This certainly won't work in general: if $Q\neq 0$, the x^{-2} term forces $a_0=0$, then the x^{-1} term forces $a_1=0$, and so on. Moreover, even if $Q=0$, if one of P and R is not zero, we often find that we are forced to have a relationship between a_0 and a_1, so even if we have one solution, we may not have two.

Therefore, while our previous method may work in special cases, it fails quite spectacularly in general. But thankfully, we know an equation with a regular singular point at $x=0$ that we *can* solve.

3.1 The Euler–Cauchy Equation

Recall the differential equation

$$x^2y''(x)+bxy'(x)+cy(x)=0, \tag{17}$$

the *Euler–Cauchy equation*, which is a particularly simple example of an equation with a regular singular point at 0. It is homogeneous in that feeding it x^α returns an x^α multiplied by a constant that depends on α,

$$\left(x^2\frac{d^2}{dx^2}+bx\frac{d}{dx}+c\right)x^\alpha=\big(\alpha(\alpha-1)+b\alpha+c\big)x^\alpha=:f(\alpha)x^\alpha. \tag{18}$$

$f(\alpha)$ is a monic quadratic function, so factorises to $(\alpha-\sigma_+)(\alpha-\sigma_-)$, where its roots are

$$\sigma_\pm=\frac{1-b\pm\sqrt{(b-1)^2-4c}}{2}. \tag{19}$$

$\sigma_\pm$ are called the *exponents* of the equation. Hence if $\sigma_+\neq\sigma_-$, two solutions to (17) are given by $x^{\sigma_+},x^{\sigma_-}$.

If $f(\alpha)$ has a double root σ, we now have $f(\alpha)=(\alpha-\sigma)^2$, and it is no longer clear how to find a second solution. It would be nice to exploit the fact that σ is a double root. But remember that since it is a double root, $f'(\sigma)=0$. This suggests that we might try differentiating

(18) with respect to α. Since the differential operator does not depend on α, we can change the order of the derivatives, so the left-hand side is

$$\left(x^2 \frac{d^2}{dx^2} + bx\frac{d}{dx} + c\right) \frac{\partial}{\partial \alpha} x^\alpha = \left(x^2 \frac{d^2}{dx^2} + bx\frac{d}{dx} + c\right) x^\alpha \log x, \tag{20}$$

while the right-hand side is

$$\frac{\partial}{\partial \alpha}\left((\alpha-\sigma)^2 x^\alpha\right) = (\alpha-\sigma)\left(2 + (\alpha-\sigma)\log x\right)x^\alpha \tag{21}$$

and hence if we put $\alpha = \sigma$, the right-hand side does indeed vanish, so a second solution is given by $\frac{\partial}{\partial\alpha} x^\alpha\big|_{\alpha=\sigma} = x^\sigma \log x$! These two solutions are linearly independent since they have different behaviour near $x = 0$.

In particular, notice that the roots of $f(\alpha)$ and their nature were key, but that it was useful for the case of two equal roots to consider the equation for more general exponents than just those that are roots of the function $f(\alpha)$, and then differentiate.

3.2 The Method of Frobenius

The form of the solutions to the Euler–Cauchy equation suggests we might have some luck if we try a series with more general powers than nonnegative integers in it.

It is beneficial to write the equation in the form

$$x^2 y''(x) + xP(x)y'(x) + Q(x)y(x) = 0, \tag{22}$$

where we have set $P(x) = xp(x)$ and $Q(x) = x^2 q(x)$, which by the definition of a regular singular point may be expanded in the power series

$$P(x) = \sum_{k=0}^{\infty} P_k x^k, \qquad Q(x) = \sum_{k=0}^{\infty} Q_k x^k. \tag{23}$$

To look for a series in powers of any sort, we should know how the operator $L := x^2 \frac{d^2}{dx^2} + xP(x)\frac{d}{dx} + Q(x)$ acts on a power of x: we see straightforwardly that

$$Lx^\alpha = \left(\alpha(\alpha-1) + \alpha P(x) + Q(x)\right)x^\alpha =: f(x,\alpha)x^\alpha, \tag{24}$$

where $f(x,\alpha)$ has a Taylor series in x, namely

$$f(x,\alpha) = \sum_{k=0}^{\infty} f_k(\alpha) x^k, \tag{25}$$

with

$$f_0(\alpha) = \alpha(\alpha-1) + \alpha P_0 + Q_0, \qquad f_k(\alpha) = \alpha P_k + Q_k. \tag{26}$$

We call $f_0(\alpha)$ the *indicial function*. This function looks like that involved in the condition on the powers in the solution to the Euler–Cauchy equation, so a natural next step is to try multiplying a power x^α by a power series and see what the operator L does to it, i.e., postulate a solution of the form

$$y_\alpha(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}, \tag{27}$$

and determine if there are conditions on α and a_k so that $Ly_\alpha(x) = 0$. We will stipulate that a_0 is the first nonzero coefficient, for reasons that will become apparent. Then

$$\begin{aligned}
Ly_\alpha(x) &= \sum_{k=0}^{\infty} a_k L x^{k+\alpha} && (28)\\
&= \sum_{k=0}^{\infty} a_k x^{k+\alpha} f(x, k+\alpha) && (29)\\
&= \sum_{k=0}^{\infty} a_k x^{k+\sigma} \sum_{j=0}^{\infty} x^j f_j(x, k+\alpha) && (30)\\
&= \sum_{m=0}^{\infty} x^{m+\sigma} \sum_{n=0}^{m} a_n f_{n-m}(\sigma+n), && (31)
\end{aligned}$$

collecting terms with the same powers of x. For this to be 0, we equate order-by-order in powers of x to find

$$\begin{aligned}
0 &= a_0 f_0(\alpha) && (32)\\
0 &= a_1 f_0(\alpha+1) + a_0 f_1(\alpha) && (33)\\
0 &= a_2 f_0(\alpha+2) + a_1 f_1(\alpha+1) + a_0 f_2(\alpha) && (34)\\
&\vdots
\end{aligned}$$

Now we see why we want $a_0 \neq 0$: it will force us to have $f_0(\alpha) = 0$, which is called the *indicial equation*. Leaving this aside for the moment,[3] we see that provided $f_0(\alpha+n) \neq 0$ for each positive integer n, given an a_0 we obtain unique expressions for all the other a_n, and by induction, these all have the form

$$a_n(\alpha) = \frac{h_n(\alpha)}{f_0(\alpha+1)\cdots f_0(\alpha+n)} a_0(\alpha), \tag{35}$$

where $h_n(\alpha)$ is a polynomial in α of degree at most n, and we have explicitly denoted the dependence of the a_k on α.[4]

Where are we up to? We have determined relations between the coefficients in the series for $y_\alpha(x)$ so that all but the lowest-order vanishes when L is applied, i.e.

$$Ly_\alpha(x) = f_0(\alpha)a_0(\alpha)x^\alpha. \tag{36}$$

Recall that the indicial equation $f_0(\alpha) = 0$ is quadratic. One might therefore expect that there are two cases: either the roots are equal or they are not. If it has two different roots, we might hope that these provide two linearly independent solutions of the differential equation. This works without modification so long as the roots do not differ by an integer, but remember that we required above that $f_0(\alpha+n) \neq 0$ to form the denominators in the a_k. Therefore there are actually three distinct cases to consider:

1. The indicial equation has two distinct roots $\sigma_\pm$, and $\sigma_+ - \sigma_-$ is not an integer.

2. The indicial equation has two equal roots σ.

3. The indicial equation has two distinct roots $\sigma_\pm$, but $\sigma_+ - \sigma_- =: s$ is a positive integer.[5]

Roots not differing by an integer We know that $f_0(\sigma_\pm + n)$ is not zero for any positive integer n, and hence the recurrence relations define one unique power series for $y_{\sigma_+}(x)$ and one for $y_{\sigma_-}(x)$. These solutions are linearly independent since they have different behaviour near $x = 0$.

Equal roots We have one root of $f_0(\alpha) = 0$, namely $\alpha = \sigma$. Again $f_0(\sigma+n) \neq 0$ for any positive integer n, so $y_\sigma(x)$ is a well-defined solution. But where do we find the other one?

Because $f_0(\alpha)$ is monic, it must have the form

$$f_0(\alpha) = (\alpha-\sigma)^2. \tag{37}$$

Ah, but we've seen this before! As previously, we consider $\partial y_\alpha/\partial\alpha$. Because L does not depend on α, we have

$$L\frac{\partial y_\alpha}{\partial\alpha}(x) = \frac{\partial}{\partial\alpha}Ly_\alpha(x) = \frac{\partial}{\partial\alpha}a_0 f_0(\alpha)x^\alpha = \frac{\partial}{\partial\alpha}(\alpha-\sigma)^2 a_0 x^\alpha = 2(\alpha-\sigma)a_0(\alpha)x^\alpha + (\alpha-\sigma)^2\big(a_0'(\alpha)x^\alpha + a_0(\alpha)x^\alpha \log x\big), \tag{38}$$

and evaluating this at $\alpha = \sigma$ gives 0. Hence $\partial y_\alpha/\partial\alpha|_{\alpha=\sigma}(x)$ is also a solution of the differential equation. What does this look like? First notice that the a_k for $k > 0$ are also functions of α, so we will again write them explicitly as $a_k(\alpha)$. Using the product rule gives

$$\left.\frac{\partial y_\alpha}{\partial\alpha}\right|_{\alpha=\sigma}(x) = \left.\frac{\partial}{\partial\alpha}\sum_{k=0}^\infty a_k(\alpha)x^{k+\alpha}\right|_{\alpha=\sigma} = \sum_{k=0}^\infty \big(a_k(\sigma)x^{k+\sigma}\log x + a_k'(\sigma)x^{k+\sigma}\big) = y_\sigma(x)\log x + \sum_{k=1}^\infty a_k'(\sigma)x^{k+\sigma}. \tag{39}$$

It is probably worth repeating that even though we now seem to have two series, the coefficients in this expression are all determined by the value of a_0. This function is also linearly independent of y_α, since it contains a logarithm. In practice $a_k'(\sigma)$ is often difficult to determine universally, so is normally tackled by matching expansions.

Roots differing by an integer Suppose the roots are σ and $\sigma - s$, where s is a positive integer. Now $f_0(\sigma - s + s) = 0$, so we need to avoid the denominator of the expression for $a_s(\sigma - s)$ becoming zero. An easy way to do this is to choose $a_0(\alpha)$ carefully: we set it to be

$$a_0(\alpha) = f_0(\alpha+1)\cdots f_0(\alpha+s)a, \tag{40}$$

where a is constant. Now none of the denominators will contain the troublesome factor $f_0(\alpha+s)$. Since $\sigma - s$ is a simple root of f_0, it also follows that

$$Ly_\alpha(x) = (\alpha-\sigma)(\alpha-\sigma+s)a_0(\alpha)t^\alpha = (\alpha-\sigma)(\alpha-\sigma+s)^2 g(\alpha)t^\alpha, \tag{41}$$

where $g(\alpha)$ is nonzero at $\alpha = \sigma$ and $\alpha = \sigma - s$. It is simple to verify as before that

$$y_\sigma(x), \qquad y_{\sigma-s}(x), \qquad \left.\frac{\partial y_\alpha}{\partial\alpha}(x)\right|_{\alpha=\sigma-s} \tag{42}$$

all vanish when L is applied to them. But they can't all be linearly independent, since there can be only two linearly independent solutions!

Indeed, we note that for $y_{\sigma-s}$, $a_0(\sigma-s) = f_0(\sigma-s+1)\cdots f_0(\sigma-1)f_0(\sigma) = 0$, which violates our condition that $a_0 \neq 0$. Indeed, $a_k(\sigma-s) = 0$ for $0 \leqslant k < s$ by a simple induction, so the series actually starts at $k = s$, i.e. the first term is proportional to t^σ. It follows that $y_{\sigma-s}(x)$ must

[3] Again, we beg the reader's indulgence in holding onto a seemingly unnecessarily general form: rest assured that this is a vital part of the Method.

[4] Somewhat dull exercise in induction: show that in fact

$$h_n(\alpha) = \sum_{\substack{k_1+\ldots+k_n=n \\ k_i \geqslant 0}} (-1)^{k_i} \prod_{m=0}^n f_{k_i}(\alpha+m).$$

[5] Without loss of generality in this case we can choose our labelling so that σ_- has the smaller real part.

be a multiple of $y_\sigma(x)$: were it not, we could form a linear combination so that the first term cancels and we are left with a series of powers of t with leading term $t^{\sigma+1}$, which is impossible since $\sigma+1$ does not satisfy the indicial equation.

On the other hand, we have $a_0'(\sigma-s) = -s f_0(\sigma-s+1)\cdots f_0(\sigma-1)$, so

$$\left.\frac{\partial y_\alpha}{\partial\alpha}(x)\right|_{\alpha=\sigma} = \sum_{k=0}^{\infty}\left(a_k'(\sigma-s)x^{k+\sigma-s} + a_k(\sigma-s)x^{k+\sigma-s}\log x\right) = h_s(\sigma-s)y_\sigma(x)\log x + \sum_{k=0}^{\infty} a_k'(\sigma-s)x^{k+\sigma-s} \tag{43}$$

by the same arguments as above. This clearly is linearly independent, since it both contains a logarithm and starts with a different power, and while the logarithmic term may vanish, $a_0'(\sigma-s)\neq 0$, so the power does not.

Hence in this case a set of linearly independent solutions is given by $y_\sigma(x)$ and $\left.\frac{\partial y_\alpha}{\partial\alpha}(x)\right|_{\alpha=\sigma-s}$. It is possible that the logarithmic part may vanish.

Putting all this together, we have shown that

Result 2. *If the differential equation (1) has a regular singular point at $x=0$,*

1. *If $f_0(\alpha)=0$ has two roots $\sigma_\pm$ that do not differ by an integer, $y_{\sigma_+}(x)$ and $y_{\sigma_-}(x)$ are two linearly independent solutions.*
2. *If $f_0(\alpha)=0$ has two equal roots σ, then two linearly independent solutions are given by*

$$y_\sigma(x) \qquad \text{and} \qquad \left.\frac{\partial y_\alpha}{\partial\alpha}(x)\right|_{\alpha=\sigma} = y_\sigma(x)\log x + \sum_{k=0}^{\infty} a_k'(\sigma)x^{k+\sigma}.$$

3. *If $f_0(\alpha)=0$ has two roots σ, $\sigma-s$, where s is a positive integer, then two linearly independent solutions are given by*

$$y_\sigma(x) \qquad \text{and} \qquad \left.\frac{\partial y_\alpha}{\partial\alpha}(x)\right|_{\alpha=\sigma-s} = h_s(\sigma-s)y_\sigma(x)\log x + \sum_{k=0}^{\infty} a_k'(\sigma-s)x^{k+\sigma-s}$$

However, this is not normally how we actually do the computations. Having arrived at the general form that such series take, we are well-equipped to derive the results algorithmically by plugging in the correct form in the first place, then finding the coefficients, rather than having to derive the form of the expansion from scratch.[6] Thankfully in many important cases, this *ad hoc* approach yields the right answers without much calculation.

Algorithm 1. *Given a differential equation of the form*

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

with a regular singular point at $x=0$, to find series solutions,

1. *Multiply by a power of x so that the coefficient of $y''(x)$ has order 2. $P(x)=xp(x)$ and $Q(x)=x^2q(x)$ are regular by definition, so we have the normal form*

$$x^2y''(x) + xP(x)y'(x) + Q(x)y(x) = 0.$$

2. *Substitute a Frobenius series solution in,*

$$y_\sigma(x) = \sum_{k=0}^{\infty} a_k x^{k+\sigma},$$

to obtain an expression of the form

$$0 = \sum_k [(k+\sigma)(k+\sigma-1) + (k+\sigma)P(x) + Q(x)]a_k x^k$$

Do not worry about the bottom limit of the series at this point: it will sort itself out later.

3. *Adjust the terms to collect powers of x so that each term in the summmand has the same power:*

$$0 = \sum_k [(k+\sigma)(k+\sigma-1)a_k + (\dots)a_{k-1} + \cdots + (\dots)a_0]x^k.$$

Continue to avoid trying to find the lowest term when doing this.

4. *Now look at the $k=0$ term to find the indicial equation.*
5. *Solve the indicial equation.*
6. *If the two roots do not differ by an integer, for each one separately, substitute it into the power series sum and equate coefficients to derive recurrence relations for the a_k.*

[6] The disadvantage of this approach is that it lacks motivation, hence this handout.

7. *If the two roots are the same, substitute this value for σ into the power series sum and equate coefficients to find the first solution $y_\sigma(x)$. Then look for coefficients b_k so that $y_\sigma(x)\log x + \sum_{k=1}^{\infty} b_k x^{k+\sigma}$ is also a solution, using the same coefficient-equating process.*

8. *If the two roots differ by an integer, substitute the larger one in as before to find y_σ. Then look for a second solution in the form $y_\sigma(x)\log x + \sum_{k=0}^{\infty} b_k x^{k+\sigma-s}$. (Note that there will be ambiguity in b_s and higher in this case since one can add a multiple of $y_\sigma(x)$ and still have a solution. One way around this is to stipulate that $b_s = 0$.)*

This is normally how solutions at regular singular points are found in practice. In the next section we will do some examples to demonstrate this.

Finally, to reassure any pure mathematicians, we have

Theorem 3 (Fuchs). *The Method of Frobenius works: that is, if p, q are analytic in the vicinity of a, then the Method produces solutions that converge on a disk with radius at least as large as the smaller of the radii of convergence of the power series of p, q.*

One proves this by estimating the coefficients using the recurrence relation and the coefficients of p, q.

4 Examples

4.1 Bessel's equation

Bessel's differential equation is

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0; \tag{44}$$

its solutions are important in various two-dimensional systems, and may be thought of as a generalisation of the trigonometrical functions. You will cover it in more detail in IB Methods, but it is useful to us here because by choosing different values for ν, it exhibits many of the phenomena we have been discussing. In monic form, it becomes

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0. \tag{45}$$

The point $x = 0$ is a regular singular point; all other points are ordinary.[7] Therefore let us focus on series expansions at 0.

The first thing to do is determine the recurrence relations and the indicial equation; this works the same whatever the value of ν. Substituting in the Frobenius series $\sum_k a_k x^k$, we have

$$0 = \sum_k \left((k+\sigma)(k+\sigma-1) + (k+\sigma) + x^2 - \nu^2\right) a_k x^k \tag{46}$$

$$= \sum_k \left(((k+\sigma)^2 - \nu^2)a_k + a_{k-2}\right) x^k. \tag{47}$$

Then the recurrence relation is

$$((k+\sigma)^2 - \nu^2)a_k + a_{k-2} = 0. \tag{48}$$

Significantly, only a_{k-2} appears, so the series will only contain terms of the form $x^{2m+\sigma}$, which will matter when we determine what sort of series we have in each case. We may therefore write

$$(2m+\sigma-\nu)(2m+\sigma+\nu)a_{2m} + a_{2(m-1)} = 0. \tag{49}$$

Putting $m = 0$ gives the indicial equation

$$(\sigma - \nu)(\sigma + \nu) = 0, \tag{50}$$

which has roots $\sigma = \pm\nu$. It is apparent that the case we are in depends on the value of ν: the difference between the roots is 2ν.

If ν is not an integer or half-integer, we have two linearly independent solutions, which have leading term $x^{\pm\nu}$. For $+\nu$, the recurrence relation is

$$a_{2m} = \frac{-1}{4m(m+\nu)} a_{2(m-1)}, \tag{51}$$

so by induction the series is given by

$$y_\nu(x) = a_0 x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\nu(\nu+1)\cdots(\nu+m)} \left(\tfrac{1}{2}x\right)^{2m} \tag{52}$$

For $-\nu$, the recurrence relation is in fact the same but with ν replaced by $-\nu$, so the other solution is simply $y_\nu(x)$ with the same replacement made.

The standard solution to Bessel's equation is given by

$$J_\nu(x) := \left(\tfrac{1}{2}z\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(\nu+m+1)} \left(\tfrac{1}{2}x\right)^{2m}, \tag{53}$$

which is called the *Bessel function of the first kind*. Γ is the Gamma-function, the "right" generalisation of the factorial to numbers that are not nonnegative integers. It satisfies $\Gamma(n) = (n-1)!$ for positive integers, and $\Gamma(z+1) = z\Gamma(z)$ for any complex z. It will be covered in full in Part II Further Complex Methods.

[7] One can also check that ∞ is an irregular singular point.

We now have to address the special cases mentioned above. If ν is a half-integer, which we can assume is positive, we might initially think that we will need to resort to a logarithmic solution, but since $m-\nu$ is never zero, in fact $y_{-\nu}(x)$ defines a solution in this case as well, with exponent $-\nu \neq \nu$, so we still have two linearly independent solutions.[8] This is therefore an example where the exponents differ by an integer but the Frobenius series produced are linearly independent.

Lastly, we have to consider the case where $\nu = n$ is an integer. Now we genuinely do have a problem: since $m-n$ can be zero, $-n$ does not produce a well-defined solution. The conventional way to deal with this is to consider the function

$$Y_\nu(x) := J_\nu(x)\cot \nu\pi - J_{-\nu}(x)\csc \nu\pi, \tag{54}$$

which is called the *Bessel function of the second kind*, or Weber's function. This exists and is linearly independent of $J_\nu(x)$ for ν not an integer. One can show using properties of the Γ-function that $J_{-n}(x) = (-1)^n J_n(x)$ for integer n (which neatly indicates the problem we are trying to solve), and it then follows that the limit as $\nu \to n$ exists, and is given by

$$Y_n(x) = \frac{1}{\pi}\frac{\partial J_\nu}{\partial \nu}(x)\bigg|_{\nu=n} + \frac{(-1)^n}{\pi}\frac{\partial J_\nu}{\partial \nu}(x)\bigg|_{\nu=-n}. \tag{55}$$

Further computation using properties of the Γ-function eventually gives

$$Y_n(x) = \frac{2}{\pi}J_n(x)(\log \tfrac{1}{2}z + \gamma) - \frac{1}{\pi}\left(\tfrac{1}{2}z\right)^{-n}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\tfrac{1}{2}z\right)^{2m} - \frac{1}{\pi}\left(\tfrac{1}{2}z\right)^{n}\sum_{m=0}^{\infty}\left(\sum_{k=1}^{m}\frac{1}{k}+\frac{1}{k+n}\right)\frac{(-1)^m}{m!(m+n)!}\left(\tfrac{1}{2}z\right)^{2m}, \tag{56}$$

where $\gamma = -\Gamma'(1)$ is the *Euler–Mascheroni constant*.

It is apparent that the computation here is pretty unpleasant, which is rather typical of such series. Our next example gives a special case where the computation can be made explicit and simple.

4.2 Using the Explicit Formula for the Second Solution

We know that given one solution $y_1(x)$ of second-order differential equation $y'' + py' + q = 0$, the other may be found using the Wronskian as

$$y_2(x) = y_1(x)\int^x \frac{W(t)}{(y_1(t))^2}\,dt = y_1(x)\int^x \frac{e^{-\int p}}{(y_1(t))^2}\,dt. \tag{57}$$

When $1/y_1^2$ has a simple explicit expression, we may be able to use this to derive an expression for the series form of a second solution. Consider the differential equation

$$y''(x) + \left(\frac{b}{x} - 1\right)y'(x) - \frac{b}{x}y(x) = 0, \tag{58}$$

where a is a fixed constant. We see that it has a regular singular point at $x = 0$. As usual, we may write this in the normal form

$$x^2y''(x) + (b-x)xy'(x) - bxy(x) = 0. \tag{59}$$

Substituting $\sum_k a_k x^{k+\sigma}$ gives

$$0 = \sum_k \big((k+\sigma)(k+\sigma-1) + (b-x)(k+\sigma) - bx\big)a_k x^{k+\sigma} \tag{60}$$

$$= \sum_k \big((k+\sigma)(k+\sigma-1)a_k + (k+\sigma)ba_k - (k+\sigma-1)a_{k-1} - ba_{k-1}\big)x^{k+\sigma} \tag{61}$$

$$= \sum_k \big((k+\sigma)(k+\sigma-1+b)a_k - (k+\sigma-1+b)a_{k-1}\big)x^{k+\sigma} \tag{62}$$

Putting $k = 0$ gives the indicial equation

$$\sigma(\sigma-1+b) = 0, \tag{63}$$

while the recurrence relation is

$$(k+\sigma-1+b)\big((k+\sigma)a_k - a_{k-1}\big) = 0 \tag{64}$$

Therefore the exponents are 0 and $1-b$. There are two cases to consider, b not an integer and b an integer. Putting $\sigma = 0$, the recurrence becomes

$$a_k = \frac{1}{k}a_{k-1}, \tag{65}$$

[8] For $\nu = 1/2$, the recurrence relation is $a_{2m} = \frac{-1}{2m(2m+1)}a_{2(m-1)}$, so a solution is given by

$$y_{1/2}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1/2}}{(2m+1)!} = \frac{\sin x}{\sqrt{x}},$$

while for $-1/2$ the recurrence relation is $a_{2m} = \frac{-1}{2m(2m-1)}a_{2(m-1)}$, so a linearly independent solution is

$$y_{-1/2}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m-1/2}}{(2m)!} = \frac{\cos x}{\sqrt{x}}.$$

This is one reason why Bessel functions may be seen as generalisations of the trigonometrical functions.

after dividing by $k-(1-b)$.[9] Of course the function with this recurrence relation is $y_1(x) = \exp x$; if we had been a little more observant, we would have noticed that if $y'' = y' = y$, (58) is solved.

Instead of considering the various cases of solutions and applying the Method of Frobenius to find the series, let us use the Wronskian. We compute using Abel's identity that the Wronskian of the equation is

$$W = \exp\left(-\int\left(\frac{b}{x}-1\right)dx\right) = Ax^{-b}e^{x}, \tag{66}$$

and so the second solution has the form

$$y_2(x) = e^{x}\int^{x} e^{-2t}t^{-b}e^{t}\,dt = e^{x}\int^{x} x^{-b}e^{-t}\,dt. \tag{67}$$

We can now expand the e^{-t} in its power series and find that

$$y_2(x) = e^{x}\int^{x}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^{k-b}\,dt = e^{x}\sum_{k=0}^{\infty}\int^{x}\frac{(-1)^k}{k!}t^{k-b}\,dt \tag{68}$$

If b is not a positive integer, everything integrates to a power of x and we find

$$y_2(x) = x^{1-b}e^{x}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1-b)}x^{k}, \tag{69}$$

which by expanding the other exponential and using some partial fractions identities we find agrees with the series

$$y_2(x) = x^{1-b}\sum_{m=0}^{\infty}\frac{1}{(1-b)(2-b)\cdots(m+1-b)}x^{m} \tag{70}$$

that the recurrence relation gives. It is apparent that such a manipulation will in general be quite difficult, so in non-degenerate cases it is usually simpler to stick to the recurrence relation. In cases where differentiation of the coefficients is difficult, this may work better.

Notice again that the difference between the exponents can be an integer without the second Frobenius series expansion ceasing to work. The special case $b=1$ is the simplest example of when the expansion fails, and for this value we compute that the second solution is

$$y_2(x) = e^{x}\int^{x}\left(\frac{1}{t}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}t^{k-1}\right)dt = e^{x}\log x + e^{x}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!k}x^{k}. \tag{71}$$

The series expansion for the latter then has coefficients

$$\sum_{k=1}^{m}\frac{(-1)^k}{k(m-k)!k!}, \tag{72}$$

which can be simplified by evaluating

$$\sum_{k=1}^{m}\frac{(-1)^k}{k(m-k)!k!} = -\frac{1}{m!}\int_0^1\frac{1-(1-t)^m}{t}\,dt = -\frac{1}{m!}\int_0^1\frac{1-u^m}{1-u}\,du = -\frac{1}{m!}\int_0^1\sum_{k=1}^{m}u^{k-1}\,du = -\frac{1}{m!}\sum_{k=1}^{m}\frac{1}{k}. \tag{73}$$

Hence in this case the second solution is

$$y_2(x) = e^{x}\log x - \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\frac{1}{k}\right)\frac{x^m}{m!}. \tag{74}$$

[9] If b is not a nonnegative integer, this is fine since this quantity never vanishes, whereas if b is a nonnegative integer, it is possible that $a_{1-b} \neq a_{-b}/(-b)$. However, the function with the recurrence relation (65) is still always a solution.