The Versatile Wronskian

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Given two differentiable functions y_1, y_2, their *Wronskian* is the determinant

$$W := \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$$

In this handout, we shall explore its uses, both in this course, and in starred sections, in later courses' material.

1 Test for Linear Independence

Recall that y_1 and y_2 are called *linearly independent on* (a,b) if there do not exist α_1 and α_2 so that $\alpha_1 y_1(x) + \alpha_2 y_2(x) = 0$ for every x in (a,b) (i.e. $\alpha_1 y_1 + \alpha_2 y_2$ is the zero function on (a,b)).

Result 1.

1. *If y_1, y_2 are linearly dependent, $W = 0$.*
2. *If $W = 0$ and $y_1(x) \neq 0$ for any $x \in (a,b)$, y_1 and y_2 are linearly dependent.*

Proof.

1. Suppose y_1 is linearly dependent on y_2. Then either $y_2(x) = 0$ for every x, in which case the result follows immediately, or we can rewrite the linear dependence condition as $y_1 = \alpha y_2$ for some constant α. But then

$$W = y_1 y_2' - y_1' y_2 = \alpha y_2 y_2' - \alpha y_2' y_2 = 0.$$

2. We note that since $y_2 \neq 0$,

$$W = y_1^2 \left(\frac{y_2}{y_1} \right)'.$$

Since $W = 0$ and $y_1 \neq 0$, we find that $(y_2/y_1)' = 0$, i.e. $y_2/y_1 = \alpha$ is constant, which as before is equivalent to $\alpha y_1 - y_2 = 0$, i.e. linear dependence. □

If we do allow $y_1(c) = 0$ for some $c \in (a,b)$, the latter result is not true. Peano gave the counterexample

$$y_1(x) := x^2, \qquad y_2(x) := x|x|.$$

One can check that y_1 and y_2 are linearly independent (there is a linear combination that vanishes for positive x, and one that is for negative x, but none for all x). On the other hand, we all know that y_1 is differentiable with $y_1'(x) = 2x$, and from the definition of derivative one finds y_2 is differentiable with $y_2'(x) = 2|x|$, so

$$W = (x|x|)(2x) - (2|x|)(x^2) = 0.$$

To summarise, the following implications hold:

$$\begin{aligned} \neg\text{LI} &\implies W = 0 \\ W \neq 0 &\implies \text{LI} \\ (\text{LI and } (\forall x \in (a,b)) y_1(x) \neq 0) &\implies W \neq 0 \\ (W = 0 \text{ and } (\forall x \in (a,b)) y_1(x) \neq 0) &\implies \neg\text{LI} \end{aligned}$$

2 Abel's Theorem

Theorem 2 (Abel's Theorem). *Suppose that y_1 and y_2 satisfy*

$$y'' + py + q = 0,$$

where p, q are also functions of x. Then

$$W' = -pW.$$

Of course it then follows that

$$W(x) = W(a) \exp\left(-\int_a^x p \right).$$

Proof. We have

$$\begin{aligned} W' &= y_1 y_2'' + y_1' y_2' - y_1' y_2' - y_1'' y_2 \\ &= y_1 y_2'' - y_1'' y_2 \\ &= (-p y_2' - q y_2) y_1 - y_2(-p y_1' - q y_1) \\ &= -p(y_1 y_2' - y_1' y_2) = -pW, \end{aligned}$$

as required. □

Corollary 3. *If p is finite and two solutions to the differential equation*

$$y'' + py' + q = 0$$

are linearly independent, they remain linearly independent.

Proof. $W(x)/W(a) = \exp\left(-\int_a^x p\right)$, so if $W(a) \neq 0$ and p is finite, $W(x) \neq 0$. □

3 Alternative to Reduction of Order

We seek the general solution of the differential equation

$$y'' + py' + q = 0. \tag{1}$$

We recall the usual procedure for reduction of order: if y_1 is a (nonzero) solution, we substitute $y = u y_1$, where u is also a function of x. Differentiating gives

$$\begin{aligned} (u y_1)' &= u' y_1 + u y_1' \\ (u y_1)'' &= u'' y_1 + 2u' y_1' + u y_1'', \end{aligned}$$

so the (1) becomes

$$\begin{aligned} 0 &= (u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + quy \\ &= y_1 u'' + (2y_1' + p y_1) u' + (y_1'' + p y_1 + q) u \\ &= y_1 u'' + (2y_1' + p y_1) u' \end{aligned}$$

since y_1 satisfies (1). This is a first-order linear equation for u', (hence *reduction of order*) so can be solved using an integrating factor, namely (after initially dividing by y_1 to make the leading coefficient 1) $y_1^2 e^{\int p}$, so

$$u'(x) = \frac{A}{y_1^2} \exp\left(-\int_a^x p \right),$$

which can be integrated again to give u as

$$u(x) = A \int_a^x \frac{1}{(y_1(t))^2} e^{-\int_a^t p} \, dt + B.$$

We can obtain an equation for a second solution a different way. Since we know that W satisfies $W' + pW = 0$, which we can always solve, we know W up to a constant factor. Therefore we can re-interpret the formula for the Wronskian as

$$y_1 y_2' - y_1' y_2 = W,$$

a first-order equation for y_2. Again, this is easy to solve using an integrating factor, so

$$\begin{aligned}\left(\frac{y_2}{y_1}\right)' &= \frac{W}{y_1^2}\\ \frac{y_2}{y_1} &= C + \int \frac{W}{y_1^2}\\ y_2 &= Cy_1 + y_1 \int \frac{W}{y_1^2}\,dt,\end{aligned}$$

where we have written the constant separately (rather than leaving it implicit in the integral) to emphasise that we may modify y_2 by adding any constant multiple of y_1. We notice that this is identical to the reduction of order solution, but the algebra was easier.

So our new procedure is

1. Solve the Abel equation $W' + pW = 0$ to find W.
2. Solve the expression used to define W, $W = y_1 y_2' - y_1' y_2$, as a first-order equation in y_2.

4 Variation of Parameters

The next stage is to produce a general way to obtain solutions to inhomogenious equations. Suppose we have found all of the solutions to the homogeneous equation (i.e. the complementary function). The big leap is that, since $Ay_1 + By_2$ satisfies the equation with A and B constant, replacing A and B by functions may give us enough extra freedom to satisfy the inhomogeneous equation.[1] Fortunately, this does turn out to be the case.

We want to solve a differential equation in the standard form

$$y'' + py' + qy = f, \tag{2}$$

where f is a function of x. Let y_1 and y_2 be two linearly independent solutions to (2). Then we expect that we can write

$$y = u_1 y_1 + u_2 y_2, \tag{3}$$

and we wish to determine u_i. There is a certain amount of freedom in choosing u_1 and u_2 since if we have uy_1y_2, it is unclear which term to put it in: we have two unknown functions, but only one equation. Therefore we specify the extra condition

$$u_1' y_1 + u_2' y_2 = 0 \tag{4}$$

to make the problem more definite; it turns out that this will be sufficient.[2] Differentiating y, we find

$$\begin{aligned}y' &= (u_1' y_1 + u_2' y_2) + u_1 y_1' + u_2 y_2' = u_1 y_1' + u_2 y_2',\\ y'' &= u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.\end{aligned}$$

Inserting this into the differential equation (2), we find

$$f = y''+py'+qy = u_1(y_1''+py_1'+qy_1)+u_2(y_2''+py_2'+qy_2)+u_1'y_1'+u_2'y_2',$$

and the first two terms cancel since y_i solve the homogeneous equation. We are left with two equations,

$$\begin{aligned}u_1' y_1 + u_2' y_2 &= 0\\ u_1' y_1' + u_2' y_2' &= f.\end{aligned}$$

But these are linear equations for u_1' and u_2', and so can be solved in exactly the same way as usual, which gives

$$u_1' = \frac{-y_2 f}{W}, \qquad u_2' = \frac{y_1 f}{W},$$

where W is once again the Wronskian. We can now find an expression for y with a single integration: the complementary function naturally emerges as the constants of integration in the u_i,

$$y = -y_1 \int \frac{y_2 f}{W} + y_2 \int \frac{y_1 f}{W}.$$

It is worth noting that this can be written using one integral, as

$$y(x) = \int_a^x \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} f(t)\,dt.$$

This will reappear in IB Methods in the theory of *Green's functions*.

5 * Higher order differential equations

If $y_1, \dots, y_n$ are differentiable functions, their Wronskian is the determinant

$$W := \begin{vmatrix} y_1 & y_2 & \cdots & y_n\\ y_1' & y_2' & \cdots & y_n'\\ \vdots & \vdots & \ddots & \vdots\\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}\end{vmatrix}.$$

This may be used to aid in the solution of higher-order differential equations, although its uses are less straightforward.

The general linear homogeneous ordinary differential equation of order n can be written as

$$y^{(n)} + \sum_{k=0}^{n-1} p_k y^{(k)} = 0 \tag{5}$$

5.1 Abel's theorem

Abel's theorem carries over with little alteration: one can show from the definition of determinant that if

$$A(t) = \begin{vmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t)\\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t)\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)\end{vmatrix} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}(t),$$

then by the product rule,

$$A'(t) = \begin{vmatrix} a_{11}'(t) & a_{12}'(t) & \cdots & a_{1n}'(t)\\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t)\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)\end{vmatrix} + \dots + \begin{vmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t)\\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t)\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1}'(t) & a_{n2}'(t) & \cdots & a_{nn}'(t)\end{vmatrix},$$

i.e. one may write the derivative as the sum of the determinants where the jth row has been replaced by its derivative.

[1] We might expect this, based on the simple examples we've done using heuristics like "multiply by another x" for constant-coefficient equations.

[2] This particular condition also turns out to be a major simplification: it will avoid us having to worry about terms containing second derivatives of the u_i.

If one applies this to the Wronskian, one finds that in all but the last term, the determinant has two rows equal and so is equal to zero. Therefore

$$W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}.$$

But if all the y_i satisfy (5), each term in the last row can be written as $-\sum_{k=0}^{n-1} p_k y^{(k)}$. Applying linearity of the determinant in the last row, we find that we again obtain n determinants, all but one of which have two rows equal and so are zero. This leaves

$$W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -p_{n-1}y_1^{(n-1)} & -p_{n-1}y_2^{(n-1)} & \cdots & -p_{n-1}y_n^{(n-1)} \end{vmatrix} = -p_{n-1}W,$$

which is of exactly the same form as Abel's theorem for second-order equations. Thus we can again solve for W as before,

$$W(x) = W(a)\exp\left(-\int_a^x p_{n-1}\right).$$

5.2 Reduction of order

Reduction of order works in a similar, but rather messier, manner: if we know y_1 solves (5), we insert uy_1 into this equation. We again find that the coefficient of u vanishes, and we have a linear equation of order $n-1$ for u'. *But this may be even harder to solve than the original equation!* In the order 2 case we were lucky that the equation was reduced to a linear first-order one, which we can always solve. But even a third-order equation may reduce to a horrible second-order equation that we can't deal with.[3] Exactly the same problem occurs with the Wronskian: it gives an order $n-1$ equation for y_n in terms of $y_1, y_2, \dots, y_n$, which is useless unless we know the rest of the y_i, and not even easy to solve. Thus the second-order case does not usefully generalise.

5.3 Variation of parameters with Cramer's rule

Contrastingly, variation of parameters extends simply to higher-order inhomogeneous differential equations: given

$$y^{(n)} + \sum_{k=0}^{n-1} p_n y^{(k)} = f,$$

we look for a solution of the form

$$y = \sum_{j=1}^{n} u_j y_j,$$

where y_j are a linearly independent set of solutions to the homogeneous equation. Now impose the extra conditions

$$\sum_{j=1}^{n} u_j' y_j^{(k)} = 0, \quad k \in \{0, 1, \dots, n-2\}$$

Differentiating the expression for y n times and applying these conditions gives

$$y^{(k)} = \sum_{j=1}^{n} u_j y_j^{(k)} \quad k \in \{0, 1, \dots, n-1\}$$
$$y^{(n)} = \sum_{j=1}^{n} u_j' y_j^{(n-1)} + u_j y_j^{(n)},$$

and substituting into the differential equation and using that y_j are solutions to the homogeneous equation causes this last to become

$$\sum_{j=1}^{n} u_j' y_j^{(n-1)} = f.$$

Hence we have a linear system of n equations

$$\sum_{j=1}^{n} u_j' y_j^{(k)} = f\delta_{kn} \quad k \in \{0, 1, \dots, n-1\},$$

which we can invert in the usual way: for example, Cramer's rule gives $u_j' = W_j/W$, where W_i is the Wronskianwith the jth column of the matrix replaced by $(0, 0, \dots, f)$. A single integration then gives the result.

6 * Integrals of solutions to Sturm–Liouville equations

The following sections will make much more sense after the second part of IB Methods.

A *Sturm–Liouville equation* is a second-order differential equation of the form

$$-(pu')' + qu = \lambda wu,$$

where $p > 0$, λ is called the *eigenvalue*, and $w > 0$ the *weight function.*

Such equations (with particular boundary conditions that force the solutions to have ~~mani interesting furry animals~~ many interesting properties) are studied in IB Methods. We are interested in two results here: firstly, in finding an expression for $\int u_1 u_2 w$, where u_1 and u_2 are solutions with corresponding distinct eigenvalues λ_1 and λ_2 respectively. Perhaps surprisingly, this is possible to do in closed form.

We begin with the equations

$$\begin{aligned} -(pu_1')' + qu_1 &= \lambda_1 wu_1 \\ -(pu_2')' + qu_2 &= \lambda_2 wu_2 \end{aligned} \tag{6}$$

satisfied by the functions. We can obtain the integral we are interested in in two ways: by multiplying the first equation by u_2 and integrating, or the second by u_1 and integrating. These give

$$-\int (pu_1')'u_2 + \int qu_1u_2 = \lambda_1 \int wu_1u_2$$
$$-\int (pu_2')'u_1 + \int qu_1u_2 = \lambda_2 \int wu_1u_2.$$

Subtracting, we notice the q terms cancel, so

$$(\lambda_1 - \lambda_2)\int u_1u_2w = -\int \left((pu_1')'u_2 - (pu_2')'u_1\right).$$

Integrating by parts gives

$$\begin{aligned} -\int \left((pu_1')'u_2 - (pu_2')'u_1\right) &= -pu_1'u_2 + pu_2'u_1 - \int (pu_1'u_2' - pu_2'u_1') \\ &= p(u_1u_2' - u_1'u_2). \end{aligned}$$

[3] Again, we see that just beyond ~~the corners of your waking mind~~ the material in this course there lie equations that cannot be solved explicitly. Most (but not all!) of the equations arising from physics are second-order, which somewhat mitigates this problem, but even a general second-order equation may not have a solution that can be written down as a simple integral. A general approach will be discussed in II Further Complex Methods, but still only applies to polynomial coefficients.

Oh hey, it's the Wronskian again! Even better, the remaining integrals have disappeared, so we have found that

$$\int u_1 u_2 w = p\frac{u_1 u_2' - u_1' u_2}{\lambda_1 - \lambda_2} = p\frac{W(u_1,u_2)}{\lambda_1-\lambda_2}. \tag{7}$$

This is also a consequence of the identity

$$u_1 L u_2 - u_2 L u_1 = -(pW(u_1,u_2))',$$

where $L = -DpD + q$ is the Sturm–Liouville operator, which holds for any u_1, u_2; it is known as *Lagrange's identity*.

A natural question is whether we can extract any information about $\int u_1^2 w$ from this. Suppose that $x \mapsto u_\lambda(x)$ solves the equation

$$-(pu_\lambda')' + qu_\lambda = \lambda w u_\lambda$$

and is differentiable in λ (and therefore also continuous). Then applying (7), we find for eigenvalues λ and $\lambda + h$

$$\begin{aligned}\int u_{\lambda+h}u_\lambda w &= p\frac{u_{\lambda+h}u_\lambda' - u_{\lambda+h}' u_\lambda}{h} \\ &= p\frac{u_{\lambda+h}u_\lambda' - u_\lambda u_\lambda' + u_\lambda' u_\lambda - u_{\lambda+h}' u_\lambda}{h} \\ &= pu_\lambda' \frac{u_{\lambda+h} - u_\lambda}{h} - pu_\lambda \frac{u_{\lambda+h}' - u_\lambda'}{h}.\end{aligned}$$

Taking the limit as $h \to 0$ gives

$$\int u_\lambda^2 w = p(u_\lambda' \partial_\lambda u_\lambda - u_\lambda \partial_\lambda u_\lambda') = pW(\partial_\lambda u_\lambda, u_\lambda),$$

which may or may not be easy to compute.

7 * Comparison Theorems

In this section we shall discuss something a bit more exciting: when do solutions of Sturm–Liouville equations have zeros?

Theorem 4 (Sturm–Picone Comparison Theorem). *Let $P, p, Q, q \colon [a,b] \rightrightarrows \mathbb{R}$ be continuous, with*

$$0 < p \leqslant P, \qquad q \leqslant Q$$

and consider the two differential equations

$$-(Py')' + Qy = 0 \tag{8}$$
$$-(py')' + qy = 0. \tag{9}$$

If u is a nonzero solution of the former with successive zeros z_1 and z_2 and v a solution of the latter, then either

1. *v has a zero in (z_1, z_2), or*
2. *there is $\alpha \in \mathbb{R}$ so that $v = \alpha u$, and then $p = P$ and $q = Q$.*

Proof. We start by quoting the following rather mysterious (but easily verified) identity due to Picone:

$$\left(\frac{u}{v}(vPu' - upv')\right)' = u(Pu')' - \frac{u^2}{v}(pv')' + (P-p)u'^2 + \frac{p}{v^2}(W(u,v))^2, \tag{10}$$

which holds for any differentiable functions with $v \neq 0$. If we now impose that u and v solve (8) and (9), this simplifies to

$$\left(\frac{u}{v}(vPu' - upv')\right)' = (Q-q)u^2 + (P-p)u'^2 + \frac{p}{v^2}(W(u,v))^2 \tag{11}$$

Suppose that v has no zeros in (z_1, z_2). Even if $v(z_1) = 0$, the limit of u/v exists at z_1 (this follows from L'Hôpital's Rule and the uniqueness theorem for linear differential equations: if v is not identically zero but $v(a) = 0$, $v'(a)$ must be nonzero). Integrating (11) over (z_1, z_2) gives

$$\left[\frac{u}{v}(vPu' - upv')\right]_{z_1}^{z_2} = \int_{z_1}^{z_2} \left((Q-q)u^2 + (P-p)u'^2 + \frac{p}{v^2}(W(u,v))^2\right).$$

Whether or not $v(z_1) = 0$ or $v(z_2) = 0$, the left-hand side evaluates to zero at both endpoints. But each term on the right is nonnegative, and the last term is strictly positive unless $v \propto u$. So if $v \not\propto u$ we obtain a contradiction, which implies that v has a zero, the first possibility.

If $v \propto u$, since $u^2, u'^2 > 0$ are continuous and p, P, q, Q are continuous the only way to ensure that the right-hand side vanishes is that $p = P$ and $q = Q$. □

Corollary 5. *Suppose $P > 0$, $Q \geqslant 0$. Then any nonzero solution to $-(Pu')' + Qu = 0$ has at most one root.*

Proof. We apply the Sturm comparison theorem using $p = P$ and $q = 0$. $v(x) = 1$ is a nonzero solution of $-(Pv')' = 0$, and has no roots. For any interval (z_1, z_2), if there is a solution u with roots at z_1 and z_2, v must have a zero in (z_1, z_2), a contradiction. □

Corollary 6 (Sturm Comparison Theorem). *Let $P, Q, q \colon [a,b] \rightrightarrows \mathbb{R}$ be continuous, with $q \leqslant Q$ and consider the two differential equations*

$$-(Py')' + Qy = 0 \tag{12}$$
$$-(Py')' + qy = 0. \tag{13}$$

If u is a nonzero solution of the former with successive zeros z_1 and z_2 and v a solution of the latter, then either

1. *v has a zero in (z_1, z_2), or*
2. *there is $\alpha \in \mathbb{R}$ so that $v = \alpha u$, and then $q = Q$.*

This is the case $p = P$ of the Sturm–Picone Comparison Theorem.

Corollary 7 (Sturm Separation Theorem). *If u, v are linearly independent solutions of*

$$-(Py')' + Qy = 0$$

and $u(z_1) = u(z_2) = 0$, then v has a zero in (z_1, z_2).

Proof. This is the $q = Q$ case of Sturm–Picone Comparison Theorem. We know that $v \neq \lambda u$, so the result follows from the other possibility in the theorem. □

Exercise Prove this directly by considering the Wronskian.

An extremely simple example of this theorem is that between any two roots of $\sin ax$ there is a root of $\cos ax$.[4] A less silly example is that the same is true of $\alpha \cos ax + \beta \sin ax$ and $\gamma \cos ax + \delta \sin ax$ for any $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma \neq 0$.[5]

Finally, we prove

Proposition 8. *Let $\lambda_1 < \lambda_2$, and suppose that u_1, u_2 solve the Sturm–Liouville equations*

$$-(pu_i')' + qu_i = \lambda_i w u_i$$

on (a,b) with the same boundary conditions $u_i(a) = u_i(b) = 0$. Then u_2 has more zeros in (a,b) than u_1.

We first note that this result does make sense: if u_1 had infinitely many zeros, there would be a point z where they accumulate, and one can then show that $u_1(z) = u_1'(z) = 0$, which again by uniqueness would force u_1 to be exactly zero.

[4] But you knew that.
[5] Although you may also have known that.

Proof. We apply the Sturm Comparison Theorem to $-(py')' + (q - \lambda_1 w)y = 0$ and $-(py')' + (q - \lambda_2 w)y = 0$. Since $\lambda_1 < \lambda_2$ and $w > 0$,

$$q - \lambda_1 w > q - \lambda_2 w,$$

and so if u_1 has n zeros in the interior of the interval, and we write them as $z_0 = a, z_1, z_2, \dots, z_{n+1} = b$, the SCT gives at least one zero in (z_{i-1}, z_i) for $i \in \{1, 2, \dots, n+1\}$, i.e. at least $n + 1$ in the interior. □

To go further would require us to use actual Sturm–Liouville theory, so we shall stop here. The theorem to remember for later is:

Theorem 9. *Let $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the eigenvalues of the Sturm–Liouville problem*

$$-(pu')' + qu = \lambda wu$$
$$u(a) = u(b) = 0.$$

with corresponding eigenfunctions u_n. Then λ_n has n zeros in (a, b).

7.1 ** Alternative explanation for Sturm–Liouville zeros

The following approach requires more theory, but provides perhaps a more intuitive picture: let $u(x, \lambda)$ be the solution to

$$-\partial_1(p\partial_1 u) + qu = \lambda u$$
$$u(a, \lambda) = 0, \qquad \partial_1(a, \lambda) = 1$$

(as the equation is linear, the value of the derivative is immaterial provided it is nonzero, so we can choose the latter for definiteness). The position of a zero z of u depends on λ: it satisfies the equation

$$u(z(\lambda), \lambda) = 0.$$

Hence we can take differentiate implicitly with respect to λ:

$$0 = z'(\lambda)\partial_1 u(z(\lambda), \lambda) + \partial_2 u(z(\lambda), \lambda).$$

Since zeros are simple, at a zero z_0 and eigenvalue λ_0, we have $u(z_0, \lambda_0) = 0$ and $\partial_1 u(z_0, \lambda_0) \neq 0$, so

$$z_0'(\lambda_0) = -\frac{\partial_2 u(z_0, \lambda_0)}{\partial_1 u(z_0, \lambda_0)}$$

Also, differentiating the differential equation with respect to λ at λ_0 gives

$$-\partial_1(p\partial_1\partial_2 u) + q\partial_2 u = w(u + \lambda_0\partial_2 u)$$

Multiplying by u and integrating with respect to x gives

$$\begin{aligned}
&\int_a^{z_0} w(u(x, \lambda_0))^2 \, dx \\
&= \int_a^{z_0} \big(- u\partial_1(p\partial_1\partial_2 u) + u(q - \lambda_0 w)\partial_2 u \big) \, dx \\
&= [-pu\partial_1\partial_2 u]_a^{z_0} + \int_a^{z_0} \big(p(\partial_1 u)(\partial_1\partial_2 u) + u(q - \lambda_0 w)\partial_2 u\big) \, dx \\
&= 0 + [(\partial_2 u)p(\partial_1 u)]_a^{z_0} + \int_a^{z_0} (\partial_2 u)\big(- \partial_1(p\partial_1 u) + (q - \lambda_0 w)u\big) \, dx \\
&= p(z_0)(\partial_2 u(z_0, \lambda_0))(\partial_1 u(z_0, \lambda_0)) + 0.
\end{aligned}$$

Plugging this into the equation for $z_0'(\lambda_0)$ gives

$$z_0'(\lambda_0) = -\frac{1}{(\partial_1 u(z_0, \lambda_0))^2} \int_a^{z_0} (u(x, \lambda_0))^2 w(x) \, dx < 0,$$

so as λ increases, the zeros of u move towards a.