Analytic Functions from Their Real Part

With No Integration

Richard Chapling*

v1 29 January 2016

Let $f : D \to \mathbb{C}$ be analytic, where D is simply connected. We shall derive a formula for f in terms of its harmonic real part. We assume initially that f is entire.

Define $\bar{f}(z) = \overline{f(\bar{z})}$. This is also an analytic function (we can show this using power series, for example).

Suppose first that x and y are real. We can write the real part of f as follows:[1]

$$2u(x, y) := f(x + iy) + \overline{f(x + iy)} = f(x + iy) + \overline{f(\overline{x - iy})} = f(x + iy) + \bar{f}(x - iy). \tag{1}$$

Now, what we would like to do is isolate a function of z only. The key is to notice that the above equation can be extended to hold for complex x and y as well as real: the right-hand side will make sense for any x and y such that $x + iy, x - iy$ are in the domain of f; in our case, this is all of $\mathbb{C}$.

Therefore, if we make x and y now independent *complex* variables, and we set

$$z = x + iy \qquad w = x - iy, \tag{2}$$

then z and w are independent complex variables.[2] Then we can invert the equations to find that

$$x = \frac{1}{2}(z + w) \qquad y = \frac{1}{2i}(z - w), \tag{3}$$

and so

$$2u\left(\frac{1}{2}(z + w), \frac{1}{2i}(z - w)\right) = f(z) + \bar{f}(w) = f(z) + \overline{f(\bar{w})}. \tag{4}$$

for any $z, w \in \mathbb{C}$. Because z and w are independent, we can set w to be a constant, say $\bar{z}_0$, and

$$f(z) = 2u\left(\frac{1}{2}(z + \bar{z}_0), \frac{1}{2i}(z - \bar{z}_0)\right) - \overline{f(z_0)}. \tag{5}$$

In a similar way, we can show that

$$f(z) = 2iv\left(\frac{1}{2}(z + \bar{z}_0), \frac{1}{2i}(z - \bar{z}_0)\right) + \overline{f(z_0)} \tag{6}$$

In particular, notice that if $z = z_0 + h$ is close to z_0, we have

$$f(z_0 + h) = 2u\left(\Re(z_0) + \frac{h}{2}, \Im(z_0) + \frac{h}{2i}\right) - \overline{f(z_0)}, \tag{7}$$

* Trinity College, Cambridge

[1] Observe that this is a concrete realisation of the decomposition of a two-dimensional harmonic function as $f(z) + g(\bar{z})$ for analytic f, g.

[2] And in particular, notice that x and y are no longer the real and imaginary parts of a single complex variable.

so, as one might expect, the harmonic function u is also evaluated near the point $(\Re(z_0), \Im(z_0)) \in \mathbb{C}^2$. With some more care regarding precisely where each variable lives, we can exploit this to prove a version for non-entire functions:

Result. *Let $f(z)$ be analytic in a neighbourhood of z_0, and let $f(x+iy) = u(x,y) + iv(x,y)$, where $x, y \in \mathbb{R}$ and $u, v : \mathbb{R}^2 \to \mathbb{R}$ are the real and imaginary parts of f. Then*

$$f(z) = 2u\left(\frac{1}{2}(z+\bar{z}_0), \frac{1}{2i}(z-\bar{z}_0)\right) - \overline{f(z_0)} \tag{8}$$

$$f(z) = 2iv\left(\frac{1}{2}(z+\bar{z}_0), \frac{1}{2i}(z-\bar{z}_0)\right) + \overline{f(z_0)} \tag{9}$$

In general, we might expect to be able to pass to a substantially larger region with analytic continuation, provided this region is simply connected: note that, for example, the function

$$u(x,y) = \frac{1}{2}\log\left(x^2 + y^2\right) \tag{10}$$

is harmonic on $\mathbb{R}^2 \setminus \{(0,0\}$. However, using the formula with $z_0 = 1$ gives

$$f(z) = \log\left(\frac{(z+1)^2}{4} + \frac{(z-1)^2}{-4}\right) - 0 = \log z, \tag{11}$$

which is only analytic on sets containing no closed loops around $z = 0$.