Elliptic Functions

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1. Periodic Functions

Recall that L is said to be a *period* of a function f if

$$f(z+L) = f(z) \tag{1}$$

for every z for which the function is defined. Obvious examples are the trigonometrical functions, e^{inz}, which all have π as a period. One has

Theorem 1 (Fourier series). *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function with period L. Then f has a* Fourier expansion,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z/L}. \tag{2}$$

We shall show that asking for a function to have two complex periods is a far stronger condition, essentially because f is determined by its behaviour on a compact region.

2. Doubly-Periodic Functions

As quite often in mathematics, we have to graft through some low-level theory before we can actually talk about the interesting stuff. For doubly-periodic functions, there are two such topics: lattices, which are essential, and convergence of various series involved, which does actually have important implications for the theory, despite just looking like analysis *qua* analysis.

2.1. Lattices

Suppose now that f has two non-parallel periods, ω_1 and ω_2, i.e

$$f(z) = f(z+\omega_1) = f(z+\omega_2). \tag{3}$$

We define a *lattice* Λ to be a module[1] over the integers given by

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 := \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}, \tag{4}$$

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[1] Remember those?

where ω_2/ω_1 is not real.[2] It turns out that all discrete translational subgroups of $\mathbb{C}$ with two independent directions have this form. Obviously Λ acts on $\mathbb{C}$ by $\omega.z = z + \omega$.

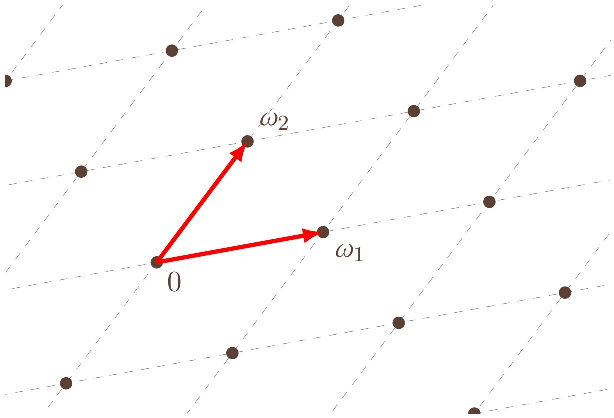
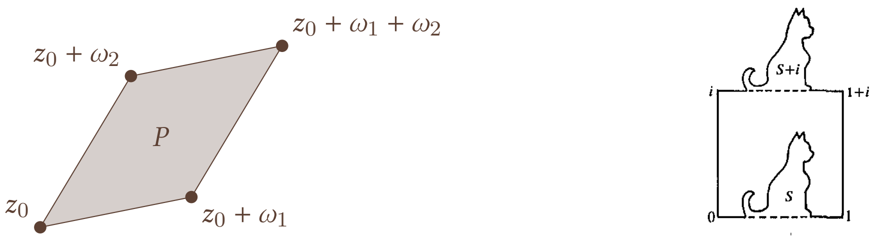


Figure 1: The lattice generated by ω_1 and ω_2

If we choose ω_1, ω_2 to be the members of Λ that have smallest modulus, then we can define a *fundamental region* for the lattice Λ as a connected, compact subset of $\mathbb{C}$ such that its translates under the action of Λ tile the plane.[3] We'll also prefer that the boundary of the region is a rectifiable Jordan curve, mainly so that we can integrate round it, but most of the time, the actual shape is not particularly important. We default to a parallelogram with sides parallel to the ω_i with lengths $|\omega_i|$, which may or may not be centred on the origin, which we call a *fundamental parallelogram*, P. (Figure 2)



(a) A fundamental parallelogram　　(b) A more general shape[4]

Figure 2: Two examples of fundamental regions

Now, two last points before we actually get to the functions: we can form the quotient of the complex plane by the action of Λ, $\mathbb{C}/\Lambda$, the set of equivalence classes of points where $z \sim z + \omega$ for $\omega \in \Lambda$. It is a straightforward computation[5] to show that $\mathbb{C}/\Lambda$ is homeomorphic to the torus T. Hence we have a natural isomorphism between functions $T \to \hat{\mathbb{C}}$ and a doubly-periodic functions on $\mathbb{C}$.

As such, it is common to abuse notions somewhat, and refer to the equivalent points $a, a + \omega, a + \omega', \ldots$ as just a (instead of the more correct $[a]$, for example). Essentially never is it important to make

[2] It's not difficult to see why one would impose this restriction: if they were both real, the ratio would either be rational or irrational. If rational, the period is actually the smallest positive value of $m\omega_1 + n\omega_2$ for integer m, n. If the ratio is irrational, the only such functions are constant.

[3] We shan't worry too much about the boundary. There are various ways round it, such as taking the fundamental region closed and saying the intersections have Jordan measure zero, or taking it open and saying that the closure of the translates gives $\mathbb{C}$.

[4] Image from Jones and Singerman, p. 67

[5] Which you may have done in Part IB Metric and Topological Spaces.

the distinction, provided that one keeps the same lattice, and notes that appropriate results are true $\pmod{\Lambda}$ (see, for example, Proposition 8, one example where this matters).

Remark 2. One final point: the quantity $\tau := \omega_2/\omega_1$ is the only important quantity related to the lattice (note that rescaling the ω_i makes no difference to it): it tells you how "thin" the torus is. It is called the *modulus* of the lattice; it can be chosen to lie in the upper half-plane. It is not difficult to show that two lattices with moduli τ, and τ' are equivalent (up to rotation and scale) if and only if there is a Möbius transformation $T(z) = (az+b)/(cz+d)$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$ with $T(\tau) = \tau'$.[6]

2.2. Elliptic Functions

Definition 3. A doubly-periodic meromorphic function is called *elliptic*.

Theorem 4 (Elliptic Liouville). *An analytic elliptic function is constant.*

Proof. Let f be analytic and elliptic. The closure of a fundamental parallelogram P is a compact set, and f is continuous, so $f(P)$ is bounded. But the range of f on $\mathbb{C}$ is identical to that of $f|_P$, so f is bounded and entire. Hence by the standard Liouville theorem, f is constant. □

Hence an elliptic function must have at least one pole. In fact, it must have at least two poles,[7] as the following result shows:

Lemma 5. *Let f be an elliptic function. Then f has finitely many poles, and if these poles are at $z = b_j$, with residues r_j, then $\sum_j r_j = 0$.*

Proof. The first part is a simple Bolzano–Weierstrass argument: if there were infinitely many, they would have an accumulation point, which cannot occur for a meromorphic function by a corollary of Laurent's theorem. For the second part, we can choose a fundamental parallelogram such that f is analytic on its boundary γ. Then

$$2\pi i \sum_j r_j = \int_\gamma f(z)\, dz = 0, \tag{5}$$

since each side of the curve γ has a corresponding side where the function has the same value (by periodicity), but traversed in the opposite direction; thus the whole thing cancels to zero. □

Therefore, if the function only has one simple pole, it has only one residue, which contradicts this lemma. Hence a nonconstant elliptic function must have at least two simple poles or one double pole.

The total order of the poles of an elliptic function contained in a fundamental region is called the *order* of the elliptic function. This actually determines far more about the function than you might expect.

Proposition 6. *Let f be an elliptic function of order N. Then f has N zeros, counted with multiplicity.*

Proof. Recall the zero-pole counting integral: for meromorphic f and closed γ,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = Z - P, \tag{6}$$

where there are Z zeros and P poles inside γ. Applying this to γ as the boundary of the fundamental region, the same argument as Lemma 5 shows that $Z - P = 0$, and the result follows. □

[6] This group of transformations is called the *modular group*, and is *extremely* important in number theory, Riemann surfaces, and group theory. See Jones and Singerman, *Complex Functions: An Algebraic and Geometric Viewpoint*, for example.

[7] As ever, poles and zeros are counted with multiplicity.

Corollary 7. *Let f be an elliptic function of order N. Then for any complex a, f takes the value a N times, counted with multiplicity.*

Proof. Apply the previous result to $g(z) = f(z) - a$, which is clearly also an elliptic function with the same poles as f. □

Exercise Prove the corresponding result for rational functions on the Riemann sphere.

Proposition 8. *Let f be elliptic, with zeros a_i, poles b_j with residues r_j Then the ordinates satisfy*

$$\sum_i a_i - \sum_j b_j \equiv 0 \pmod{\Lambda}. \tag{7}$$

Proof. Take γ to be the border of a fundamental parallelogram, and consider the integral

$$\frac{1}{2\pi i}\int_\gamma z\frac{f'(z)}{f(z)}\,dz. \tag{8}$$

This is clearly equal to the left-hand side of (7). Now, on opposite sides of the parallelogram we have

$$\frac{1}{2\pi i}\int_{z_0+\omega_1}^{z_0+\omega_2+\omega_1} z\frac{f'(z)}{f(z)}\,dz - \frac{1}{2\pi i}\int_{z_0}^{z_0+\omega_2} z\frac{f'(z)}{f(z)}\,dz = \frac{\omega_1}{2\pi i}\int_{z_0}^{z_0+\omega_2} \frac{f'(z)}{f(z)}\,dz \tag{9}$$

The trick that is required here is to notice that the integral on the right-hand side, which we recognise as $[\log f(z)]_{z_0}^{z_0+\omega_2}$, must change by an integer multiple of 2π since $f(z+\omega_2) = f(z)$, so the curve maps to one which encircles the origin an integer number of times, and closes. Hence the right-hand side of this last equation is an integer multiple of ω_1. Applying this to the other pair of sides gives an integer multiple of ω_2, and the result follows. □

This is basically the limit of the general theory we can consider using complex analysis; we now construct the most important elliptic function, from which we can subsequently obtain a more constructive theory.

3. The Weierstrass Functions

In this section we define a particular elliptic function, deduce some of its properties, and show that it is central in the theory, in that all other elliptic functions may be expressed in as rational functions of it.

3.1. Definition, basic properties, differential equation

Definition 9 (Weierstrass pe-function). The Weierstrass elliptic function associated with a lattice Λ is

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda}{}' \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}, \tag{10}$$

where the prime on the sum shall denote that the sum excludes terms with zero denominator.[8]

[8] This probably looks rather unintuitive to you: why would you consider such a thing? See Appendix A for an explanation.

Remark 10. Remember that in general, the order of summation in a double series matters: summing over rows, then columns, may be different from summing over columns then rows. And both might be different from taking a "spiral", which covers the points as a succession of squares around $m = n = 0$, say. However, this double series is actually absolutely convergent (you can check that for a particular z away from the lattice points, the sum is bounded above by $A \sum'_{\omega\in\Lambda} |\omega|^{-3}$, which is bounded above by considering how many lattice points there are near each circle (i.e., about $2\pi r$), each with modulus about r^{-3}, so the total is bounded by a multiple of the sum of the reciprocal squares, which is finite).[9] Hence the order of summation of the series does not matter.

Remark 11. However, the above is not enough to verify that we have a meromorphic function. We need that if a sequence of meromorphic functions f_n converges uniformly on compact sets to a function f, then f is meromorphic.[10] In particular, one can check this for the definition of $\wp$ if we take compact subsets of $\mathbb{C} \setminus \Lambda$, so it follows that $\wp$ and its derivatives are meromorphic functions.

Lemma 12 (Basic properties of $\wp$). *$\wp$ is an even elliptic function with a double pole at each lattice point.*

Proof. The above Remarks deal with the technical equipment necessary to show that $\wp$ is meromorphic. It is also clear from this that $\wp$ has double poles at each lattic point, and nowhere else. Since the series can be rearranged and $(-z-\omega)^2 = (z-(-\omega))^2$, $\wp$ is even (the terms in each series are identical). The sticky part is the periodicity. If we differentiate, we find that

$$\wp'(z) = -2\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3} \tag{11}$$

(notice that there is no prime on this summation!). Clearly for $\wp'(z)$, replacing z by $z+\omega_0$, $\omega_0 \in \Lambda$ merely permutes the terms in the series, and hence $\wp'$ is an elliptic function. Therefore $\wp'(z+\omega_0) - \wp'(z) = 0$ identically, and so $\wp(z+\omega_0) - \wp(z) = c_{\omega_0}$. If we put $z = -\omega_0/2$, we find that

$$c_{\omega_0} = \wp(\omega_0/2) - \wp(-\omega_0/2) = 0, \tag{12}$$

since $\wp$ is even. Hence $\wp$ is doubly-periodic, and so elliptic. □

Defining the *modular invariants* of the lattice as

$$g_2 = 60 {\sum_{\omega\in\Lambda}}' \frac{1}{\omega^4}, \qquad g_3 = 140 {\sum_{\omega\in\Lambda}}' \frac{1}{\omega^6}, \tag{13}$$

we can derive the first few terms of the Laurent series of $\wp(z)$ at $z = 0$:

$$\wp(z) = \frac{1}{z^2} + {\sum_{\omega\in\Lambda}}' \frac{1}{\omega^2}\left(\left(1-\frac{z}{\omega}\right)^{-2} - 1\right) \tag{14}$$

$$= \frac{1}{z^2} + {\sum_{\omega\in\Lambda}}' \sum_{n=1}^{\infty} \frac{1}{\omega^{n+2}}(n+1)z^n \tag{15}$$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} z^n(n+1) {\sum_{\omega\in\Lambda}}' \frac{1}{\omega^{n+2}} \tag{16}$$

$$= \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + O(z^6), \tag{17}$$

where the order of summation may be interchanged because the series converge uniformly on a compact neighbourhood of the origin. The odd terms vanish because $-\omega$ and ω are both in the lattice, so raising both to an odd power causes them to cancel in the sum; we again find that $\wp$ is even.

[9] Obviously this is not actually a proof, but it does indicate the method used.

[10] A proof of this can be found in Jones and Singerman, *Complex functions: An Algebraic and Geometric Viewpoint*, §3.7

Corollary 13 (Differential equation for $\wp$). *$\wp$ and its derivative satisfy the following nonlinear differential equation:*

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3. \tag{18}$$

Proof. This proof is mainly tedious computation. Recall that elliptic functions with no poles are constant; hence it suffices to check that

$$f := \wp'^2 - 4\wp^3 + g_2\wp + g_3 \tag{19}$$

has no poles, and is zero somewhere. The only place $\wp$, and hence $\wp'$, can have poles is at lattice points, of which we need only check 0 by periodicity. We have as $z \to 0$:

$$\wp'(z) = -\frac{2}{z^3} + \frac{g_2}{10}z + \frac{g_3}{7}z^3 + O(z^5) \tag{20}$$

$$(\wp'(z))^2 = \frac{4}{z^6} - \frac{2g_2}{5}\frac{1}{z^2} - \frac{4g_3}{7} + O(z^2) \tag{21}$$

$$(\wp(z))^3 = \frac{1}{z^6} + \frac{3g_2}{20}\frac{1}{z^2} + \frac{3g_3}{28} + O(z^2), \tag{22}$$

and so

$$f(z) = (4-4)\frac{1}{z^6} + \left(-\frac{2g_2}{5} - \frac{3g_2}{5} + g_2\right)\frac{1}{z^2} + \left(-\frac{4g_3}{7} - \frac{3g_3}{7} + g_3\right) + O(z^2) = O(z^2), \tag{23}$$

and hence f has no poles and tends to 0 at $z = 0$, so it must be identically zero. □

Corollary 14. *$\wp$ is the inverse of the elliptic integral*

$$\int^z \frac{dw}{\sqrt{w^3 - g_2 w - g_3}}. \tag{24}$$

3.2. Zeros of $\wp$ and $\wp'$

Zeros of $\wp$ itself are difficult to find analytically. On the other hand, we have

Lemma 15. *The function $\wp(z) - \wp(a)$ has precisely two roots in a fundamental parallelogram: $z = \pm a$.*

Proof. Clearly $z = a$ is a root. Also, $\wp$ is even, so $\wp(-a) - \wp(a) = 0$, so $z = -a$ is a root. Since $\wp$ has order 2, these are the only two roots. □

This will be useful later on, to represent a general elliptic function. For $\wp'$, the situation is somewhat simpler:

Lemma 16. *$\wp'(z)$ has three distinct roots in a fundamental parallelogram, at $z = \omega_i$, where we set $\omega_3 = -(\omega_1 + \omega_2)$. If we set $e_i = \wp(\omega_i/2)$, the e_i are distinct.*

Proof. $\wp'$ has order 3, so it must have precisely three zeros in a fundamental parallelogram. Since $\wp'$ is odd and $-\omega_i$ is a period, we have

$$\wp'(z) = \wp'(z - \omega_i) = -\wp'(\omega_i - z), \tag{25}$$

and putting $z = \omega_i/2$ gives the first part. For the second part, notice that the function $f(z) := \wp(z) - e_i$ has at least double root at $z = \omega_i/2$, since $f(\omega_i/2) = f'(\omega_i/2) = 0$. But since f has order 2, we must have only a double root at ω_i, and f has no more roots in a fundamental parallelogram, so $f(\omega_j/2) = e_j - e_i \neq 0$ for $i \neq j$. □

Hence we have the factorisation of (18) as

$$\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \tag{26}$$

Using Vietà's formulae, we then obviously have

$$e_1 + e_2 + e_3 = 0 \tag{27}$$

$$e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{4}g_2 \tag{28}$$

$$e_1e_2e_3 = \frac{1}{4}g_3. \tag{29}$$

3.3. Addition and duplication formulae

Theorem 17. *Let $u + v + w = 0$, with none of u, v, w equivalent to $0 \pmod{\Lambda}$. Then*

$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{vmatrix} = 0. \tag{30}$$

This is often written with $u = z_1$, $v = z_2$, $w = -(z_1 + z_2)$.[11]

Proof. Consider the elliptic function

$$f(z) = \wp'(z) - A\wp(z) - B. \tag{31}$$

We can choose A and B so that $f(u) = 0 = f(v)$. (these equations determine A and B unless $u = \pm v$ by considering the determinant of the system in A, B and applying Lemma 15). Now, f has order 3, so it must have exactly three zeros in a period. The pole is at zero, so the zeros must sum to 0. We defined f to have u and v as zeros, and hence $-u - v = w$ is the other, and so we have the three equations

$$\wp'(u) = A\wp(u) + B \tag{32}$$

$$\wp'(v) = A\wp(v) + B \tag{33}$$

$$\wp'(w) = A\wp(w) + B. \tag{34}$$

Hence the points $(\wp(u), \wp'(u))$, $(\wp(v), \wp'(v))$ and $(\wp(w), \wp'(w))$ all lie on the line $y = Ax + B$, so the given determinant is zero as required. □

We can in fact prove a form of the addition theorem that does not contain $\wp'(u + v)$: we do this by evaluating A in the previous proof in two different ways.

Theorem 18 (Addition theorem for $\wp$).

$$\wp(u + v) = \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2 - \wp(u) - \wp(v). \tag{35}$$

[11] This is such a 19th-century result: determinants, explicit functions, what more could one want?

Proof. We have

$$\wp'(z) = -A\wp(z) + B, \tag{36}$$

when z is any of $u, v, -u-v$, but from the differential equation, we also have that these points all satisfy

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{37}$$

so it suffices to find the points of intersection of the cubic $y^2 = 4x^3 - g_2x - g_3$ and the line $y = Ax + B$. Substituting in, we find that

$$0 = 4x^3 - A^2x^2 + (2AB - g_2)x + (B - g_3). \tag{38}$$

We know the three roots of this equation are $\wp(u)$, $\wp(v)$ and $\wp(-u-v) = \wp(u+v)$, and applying Vietà's formula gives

$$\wp(u) + \wp(v) + \wp(u+v) = \frac{1}{4}A^2. \tag{39}$$

Meanwhile, solving the original two equations that we had for f gives

$$A = \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}, \tag{40}$$

and putting these last two equations together gives the result. □

Remark 19. A shocking general result due to Weierstrass is that any meromorphic function f which possess an algebraic addition theorem (that is, a nontrivial equation $G(f(u), f(v), f(u+v)) = 0$, where G is an algebraic function) is either

1. an algebraic function of z,
2. an algebraic function of $e^{i\pi z/\omega}$, where ω is an appropriate constant.
3. an algebraic function of $\wp(z)$, with suitably chosen periods.[12]

This provides yet another reason to study elliptic functions!

3.4. Representation results

We actually now have enough equipment to prove

Theorem 20 (Representation by $\wp$ and $\wp'$).

1. *Any even elliptic function is expressible in the form $f(z) = R(\wp(z))$, where R is a rational function.*
2. *Any elliptic function is expressible in the form $f(z) = R(\wp(z)) + S(\wp(z))\wp'(z)$*

Proof. 1. The result is trivial if f is constant, so suppose that f has order $N > 0$ (recall that $f(z) = c$ thus has N solutions with multiplicity, and in particular, finitely many). The temptation is to dive straight in and try to kill all f's poles with Weierstrass functions. The problem with doing this is that things get a bit hairy when some of these poles are at the the lattice points or the $\omega_i/2$. To get around this, we choose c and d so that $f(z) = c$ and $f(z) = d$ have only simple roots, none of which are equivalent to the bad points mentioned above. (Since $f'(z) = 0$ only occurs on a discrete set, there are only finitely many points to avoid, so this is easily done.) Since f is

[12] The standard reference for this is Forsyth's *Theory of Functions of a Complex Variable*, Chapter XIII, which you can find for free on `http://archive.org/details/theoryoffunction028777mbp`, for example.

even, the simple, nonequivalent roots of $f(z) = c$ form a set $\{a_1, -a_1, a_2, -a_2, \dots, a_n, -a_n\}$, and those of $f(z) = d$ form a similar set $\{b_1, -b_1 \dots, b_n, -b_n\}$. Thus

$$g(z) = \frac{f(z) - c}{f(z) - d} \tag{41}$$

has simple zeros at $\pm a_i$ and simple poles at $\pm b_i$ for each i. As we showed in Lemma 15, $\wp(z) - \wp(a)$ has simple zeros only at $\pm a$, so the elliptic function

$$g(z) \prod_{i=1}^{n} \frac{\wp(z) - \wp(b_i)}{\wp(z) - \wp(a_i)} \tag{42}$$

has no poles or zeros. It follows that it is constant, and hence

$$\frac{f(z) - c}{f(z) - d} = A \prod_{i=1}^{n} \frac{\wp(z) - \wp(a_i)}{\wp(z) - \wp(b_i)} \tag{43}$$

for some constant A; we can then solve this equation for f to obtain the expression for f as a rational function of $\wp$.

2. Fortunately this is very simple: for any function f, we can decompose f into

$$f(z) = \left(\frac{f(z) + f(-z)}{2} \right) + \wp'(z) \left(\frac{f(z) - f(-z)}{2\wp'(z)} \right); \tag{44}$$

the first bracket is obviously an even elliptic function, and the second bracket is a quotient of two odd elliptic functions, so is even and elliptic. We can then use the first part of the theorem on both parts to obtain the result.

□

A. Why the Weierstrass function?

When one is used to starting the theory of trigonometrical functions with $\sin z$ and $\sin' z = \cos z$, basing the theory of elliptic functions on $\wp$ may seem a bit strange. We can actually construct $\wp$ using an argument similar to trigonometric functions: recall that $\sin \pi z$ has the infinite product representation

$$\sin \pi z = S(z) := z \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n}\right) e^{z/n}, \tag{45}$$

by Weierstrass's theorem. The product converges uniformly and absolutely due to the exponential factor, although this does make it difficult to verify periodicity. Since infinite products are normally more difficult to work with, we can try taking a logarithmic derivative, which gives

$$Z(z) := \frac{S'(z)}{S(z)} = \pi \cot \pi z = \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' \frac{1}{z - n} + \frac{1}{n} \tag{46}$$

This is the prototypical periodic function with simple poles: it has a simple pole at each integer with residue 1, as you know from using it to calculate sums with contour integration; it is also odd. Again, the periodicity is not obvious, but we can consider its derivative:

$$P(z) := -Z'(z) = \pi^2 \operatorname{cosec}^2 \pi z = \frac{1}{z^2} + \sum_{n=-\infty}^{\infty}{}' \frac{1}{(z-n)^2} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}, \tag{47}$$

which is even, has a double pole at each integer, and is clearly periodic. We can derive a differential equation satisfied by P using the power series: it has the Laurent series expansion

$$P(z) = \frac{1}{z^2} + \sum_{n\in\mathbb{Z}}{}' \sum_{k=0}^{\infty} \frac{(k+1)}{n^{k+2}} z^k = \frac{1}{z^2} + \sum_{n\in\mathbb{Z}}{}' \frac{1}{n^2} + 3z^2 \sum_{n\in\mathbb{Z}}{}' \frac{1}{n^4} + 5z^4 \sum_{n\in\mathbb{Z}}{}' \frac{1}{n^6} + O(z^6), \tag{48}$$

as with $\wp$. If we define $H_n = \sum{}' n^{-n}$, we can write this as

$$P(z) = \frac{1}{z^2} + H_2 + 3H_4 z^2 + 5H_6 z^4 + O(z^4) \tag{49}$$

Then we find that the function

$$f = P'^2 - 4P^3 + 12H_2 P = O(z^2) \tag{50}$$

as $z \to 0$, provided that $H_2^2 = 5H_4$ and $2H_2^3 = 35H_6$. This turns out to be true, using the explicit formula for P. Moreover, one can show that $P(z), P'(z) \to 0$ as $|\Im(z)| \to \infty$, so f is analytic and bounded on a fundamental strip, $0 < \Re(z) < 1$. Hence f is constant, and the calculation shows that it is zero, so P satisfies the nonlinear differential equation

$$P'^2 = 4P^3 - 12H_2 P. \tag{51}$$

Remark 21. The vanishing of f gives infinitely many relations between the numbers H_{2k}, which are of course the numbers $2\zeta(2k)$, the values of the Riemann zeta-function at positive even integers. Indeed, we find that any one of these is a rational multiple of a power of $H_2 = \pi^2/3$.

This looks silly and trivial when written like this, but if you think for a second, you'll notice that *exactly the same idea works for* $\wp$, giving infinitely many relations between the lattice constants $G_{2k} = \sum{}'_{\omega\in\Lambda} \omega^{-2k}$ that appear in the Laurent series of $\wp$. We discover these can all be written in terms of the first two, G_4 and G_6.[13]

We can follow an analogous process in defining $\wp$: instead of S, we construct an entire function σ so that it has zeros at every lattice point:

$$\sigma(z) = z \prod_{\omega\in\Lambda}{}' \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right), \tag{52}$$

the *Weierstrass sigma-function*

This is constructed in exactly the same way as the sine product. Sadly, since this is entire, it cannot be elliptic, and we'll have to work harder to get an elliptic function out of it. Taking the logarithmic derivative gives

$$\zeta(z) := \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega\in\Lambda}{}' \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega}, \tag{53}$$

which is an odd function that has simple poles att[14] each lattice point. This is the *Weierstrass zeta-function*, not to be confused with Riemann's![15] As we know, this *still* can't be elliptic, because it only has one pole in a fundamental region. However, if we differentiate again, we find

$$-\zeta'(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda}{}' \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \wp(z), \tag{54}$$

[13] This is the slightest hint of the beginning of the wonderful theory of *modular forms*, where all sorts of ridiculous identities like this can be proven in similarly trivial ways.

[14] at and only at

[15] This is another reason for using s as the argument of the Riemann zeta-function.

which is our old friend with double poles at each lattice point, and really is elliptic. Hence the theory can be constructed parallel to a development in trigonometrical functions, but to do so requires more machinery (e.g., infinite products) than it seemed sensible to exposite in an introduction of this length.

ζ and σ have their own nice properties: one can show that $\zeta(z+\omega_i)-\zeta(z)$ are nonzero constants, and from there derive the multiplicative properties of $\sigma(z+\omega_i)/\sigma(z)$. Such functions are called quasi-periodic, and there many useful examples: the theta-functions of Jacobi are probably the most important.

ζ does have a number of uses of its own: one can use it to construct elliptic functions with simple poles (it turns out that the conditions on the residues, poles and zeros we gave in the second section above are all that is required): for example, it is easy to check that $\zeta(z+a)-\zeta(z-a)$ is elliptic, and in fact is non-constant if a is not a period. Quotients of σ can be used to construct elliptic functions, but this is also beyond the scope of what we consider here. This is also probably the place to point out that it was precisely the annoying exponential factors that we had to put in to get σ to converge which are the bête noire of ζ's periodicity: both parts of the theory conspire to consistency here.

The Jacobi theta-functions give an alternative construction of elliptic functions, producing as the basic functions ones with simple poles. From a numerical point of view, the theta-functions have very fast convergence, and so are invaluable for actually drawing pictures and doing gritty realistic calculations with elliptic functions.[16] They also have number-theoretic importance relating to the divisor functions.

Bibliography

There are loads of books on elliptic functions,[17] but the author consulted the following in preparing this document.

1. Whittaker, E.T. and Watson, G.N., *A Course of Modern Analysis* (CUP 1927), esp. Chap. XX.
   The classic special functions book. Beware that the conventions are no longer standard, but if you only use one reference work, it should probably be this one. Also covers the other version of elliptic functions, using the Jacobi theta-functions, which we have not discussed, and elliptic integrals.

2. Jones, G.A. and Singerman, D., *Complex Functions: An Algebraic and Geometric Viewpoint* (CUP 1987), esp. Ch. 3.
   This was the book that made me fall in love with elliptic functions. Covers all the standard Weierstrass material in a simple way, but also has all the convergence details.

3. McKean, H. and Moll, V., *Elliptic Curves: Function Theory, Geometry, Arithmetic* (CUP 1999).
   Ch. 2 is about elliptic functions, but takes quite a different approach from the usual texts, starting with the elliptic integrals, and only much later discussing the functions ab initio. *It also discusses the functions as conformal maps, the arithmetic–geometric mean, and some discussion of hyper-elliptic integrals, which are far more complicated. (It does, however, have a rather quaint habit of writing* $\sqrt{-1}$ *all over the place.)*

4. Riemann, B., *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (Inauguraldissertation, Göttingen, 1851), and
   Theorie der Abel'schen Functionen (Journal für die reine und angewandte Mathematik, 54 (1857), 101-155).
   (Both are available in translation on the Internet.) I plead some partiality here, but these are Riemann's most

[16] Many of the fast formulae for calculating π come from theta-functions.

[17] Most of the more advanced complex function theory books include a chapter, purely because the theory is so quick to derive with a fairly minimal set of equipment.

important papers in complex analysis:[18] *Riemann creates the concept of Riemann surface to solve the problem of understanding* Abelian functions, *which one can consider as the inverses of* Abelian integrals, *which are integrals of arbitrarily complicated algebraic expressions (elliptic integrals being a very special case): his new surfaces are precisely the natural home for this theory. The theory of Abelian functions was, according to Klein, "the indisputable pinnacle of mathematics" when he was a student during the First World War, so you probably owe it to yourself to learn something about them.*

5. Forsyth, A.R., *Theory of Functions of a Complex Variable* (CUP 1918, reprinted in two volumes by Dover, 1965).
   Chapter XIII details the proof of Weierstrass's theorem about functions with an addition theorem.

6. Schwalm, W., *Elliptic Functions sn, cn, dn, as Trigonometry* (`http://www.und.edu/instruct/schwalm/MAA_Presentation_10-02/handout.pdf`).
   A nice introduction to the basics of the trigonometric version of the derivation of elliptic functions and their properties. There is presumably a book with this theory in it, but it remains so far undiscovered.

[18] Counting the ζ-function paper as number theory for convenience.