The Hypergeometric Function and the Papperitz Equation

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v1 9 March 2016

1 Second-Order Differential Equations with Three Regular Singular Points

1.1 Summary of previous theory

We start with a general second-order differential equation:

$$u''(z) + p(z)u'(z) + q(z) = 0, \tag{1}$$

where p and q are in general meromorphic functions. Recall that at $z = a$, this equation has an

- *ordinary point* if $p(z)$ and $q(z)$ are analytic at a,
- *regular singular point* if $p(z)$ or $q(z)$ is not analytic at a, but $(z-a)p(z)$ and $(z-a)^2q(z)$ are analytic at a. (I.e., p has a simple pole at a or q has a simple or double pole at a.)
- *irregular singular point* if $(z - a)p(z)$ and $(z - a)^2q(z)$ are not analytic at a (i.e. p and q have higher-order poles).

We've all seen that we often want to talk about the point at ∞ as well. This works slightly differently: we do the usual trick of moving to 0 using the substitution $t = 1/z$. Then

$$\frac{du}{dx} = \frac{dt}{dx}\frac{du}{dt} = -\frac{1}{x^2}\frac{du}{dt} = -t^2\frac{du}{dt} \tag{2}$$

$$\frac{d^2u}{dx^2} = \frac{2}{x^3}\frac{du}{dt} - \frac{1}{x^4}\frac{d^2u}{dt^2} = 2t^3\frac{du}{dt} - t^4\frac{d^2u}{dt^2}, \tag{3}$$

and the differential equation (1) becomes

$$\frac{d^2u}{dt^2} + \left(-\frac{p(1/t)}{t^2} + \frac{2}{t}\right)\frac{du}{dt} + \frac{q(1/t)}{t^4}u = 0. \tag{4}$$

Therefore the nature of the point at ∞ can be examined by considering the nature of the singularities of $p(1/t)/t^2 - 2/t$ and $q(1/t)/t^4$ at $t = 0$. Equivalently, $z^2p(z) - 2z$ and $z^4q(z)$ should be bounded as $z \to \infty$ for an ordinary point and $O(z)$ as $z \to \infty$ for a regular singular point.

If a differential equation has only ordinary and regular singular points on $\hat{\mathbb{C}}$, then it is called a *Fuchsian differential equation*.

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Finding solutions at ordinary points is easy: u must be analytic, so we can simply substitute a Taylor series in and compute order-by-order using recurrence relations from the expansions of p and q.

To find solutions at regular singular points, recall the *method of Frobenius*: we try a series of the form

$$u(z) = \sum_{n=0} a_n (z-a)^{n+\sigma}, \tag{5}$$

where σ is to be determined. If

$$p(z) = \frac{B}{z-a} + O(1), \qquad q(z) = \frac{C}{(z-a)^2} + O\left(\frac{1}{z-a}\right) \qquad \text{as } z \to a, \tag{6}$$

then we can find the *indicial equation* that σ has to satisfy, by considering the bottommost terms of the series:

$$0 = \sigma(\sigma-1)(z-a)^{\sigma-2} + B\sigma(z-a)^{\sigma-2} + C(z-a)^{\sigma-2},$$

or

$$\sigma^2 + (B-1)\sigma + C = 0. \tag{7}$$

This has two roots, α and α', that satisfy

$$\alpha + \alpha' = 1 - B, \qquad \alpha\alpha' = C \tag{8}$$

by Vietà's formulae; we call these the *exponents* of the differential equation at the regular singular point a.[1]

1.2 Three Regular Singular Points

It turns out that if the equation only has three regular singular points, the solutions to the equation are easy to describe in generality; since this case actually encompasses many of the equations we encounter in physics (such as Bessel's equation, Legendre's equation, the Airy equation, and so on), it is well worth discussing this case in detail. If p and p have only finitely many singularities, they must be rational functions, and in particular, if the singular points are to be at a, b, c, with exponents $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ respectively, then p must take the form

$$p(z) = \frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} + P(z), \tag{9}$$

where $P(z)$ is entire. We obviously need $p(z) \to 0$ as $z \to \infty$, or we cannot hope for a regular singular point. Hence $P(z) \to 0$, so it is bounded, and so $\equiv 0$ by Liouville's theorem. Also, we find that the coefficient of z in $z^2 p(z) - 2z$ is

$$1 - \alpha - \alpha' + 1 - \beta - \beta' + 1 - \gamma - \gamma' - 2, \tag{10}$$

which we also need to vanish to have an ordinary point, so we have the condition

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1. \tag{11}$$

There is now the question of finding q. We know that it should look like $\alpha\alpha'/(z-a)^2$ near $z = a$, with obvious parallel expressions near b and c, so a natural decomposition is

$$q(z) = \frac{k_a z + l_a}{(z-a)^2} + \frac{k_b z + l_b}{(z-b)^2} + \frac{k_c z + l_c}{(z-c)^2} + (\text{entire}). \tag{12}$$

[1] And this calculation shows precisely why we need the conditions that we specified for a solution of this form to exist: otherwise, the indicial equation wouldn't join up the bottom of the series correctly.

Combining the fractions gives

$$q(z) = \frac{Q_1(z)}{(z-a)^2(z-b)^2(z-c)^2} + Q_2(z), \tag{13}$$

where Q_1 is at most a quintic polynomial and Q_2 is entire. We need $z^4 q(z)$ bounded as $z \to \infty$, so $Q_2(z) \to 0$ and Liouville again tells us that $Q_2(z) \equiv 0$. We also see that $Q_1(z)$ must actually only be a quadratic. Putting all this information together shows that we can write q in the form

$$q(z) = \frac{1}{(z-a)(z-b)(z-c)} \left(\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c} \right). \tag{14}$$

Therefore the most general differential equation with specified singular points and exponents is

$$\begin{aligned}\frac{d^2u}{dz^2} &+ \frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\frac{du}{dz} \\ &+ \frac{1}{(z-a)(z-b)(z-c)} \left(\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c} \right) u = 0, \end{aligned} \tag{15}$$

which is known as the *Papperitz equation.*[2] As shown above, the exponents must satisfy

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1, \tag{16}$$

but this is effectively the only restriction on the parameters in the equation. The form of the differential equation should be preserved by Möbius transformations, since they effectively just move the singular points around; it is a tedious calculation (Exercise!) to show that this is in fact the case.

1.3 Riemann's P-symbol

Many years before Papperitz wrote his equation down, Riemann considered solutions to a differential equation with three regular singular points, without actually writing it down. Being Riemann, he came up with a unique way of describing the solutions: we write

$$u = P \left\{ \begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\} \tag{17}$$

to mean that u is a solution of (15). The monstrosity on the right-hand side is called *Riemann's P-symbol.*[3,4] We can think of P as representing the equation and its solution-set as appropriate, since it determines both.

Notice that if u is a solution given by (17), then

$$u_1 = \left(\frac{z-a}{z-b}\right)^k \left(\frac{z-c}{z-b}\right)^l u \tag{18}$$

[2]First apparently written down by Papperitz in *Math. Ann.* XXV. (1885), p. 213.

[3]Following Riemann, we shall be deliberately vague about what this symbol actually is: its use lies in its manipulation as a formal bookkeeping object, rather than its exact nature.

[4]Sometimes this is called the Papperitz symbol, presumably because it's written with a P, but this is an absurd attribution: Riemann's paper in *Abh. d. k. Ges. d. Wiss. zu Göttingen*, VII, where he introduced this notation was published in 1857, the same year that Papperitz was born!

also satisfies a differential equation with three regular singular points at a, b, c, but with differing exponents. Because the Papperitz differential equation is uniquely determined by the positions of the singular points and their exponents, we can immediately write down

$$\left(\frac{z-a}{z-b}\right)^k \left(\frac{z-c}{z-b}\right)^l P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} a & b & c & \\ \alpha+k & \beta-k-l & \gamma+l & z \\ \alpha'+k & \beta'-k-l & \gamma'+l & \end{Bmatrix}; \tag{19}$$

this is one of the two most important transformation laws; it changes the exponents, but not the singular points.

On the other hand, Möbius transformations move the singular points but do not change the exponents. This gives us the other important transformation: if T is a Möbius transformation, then

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} T(a) & T(b) & T(c) & \\ \alpha & \beta & \gamma & T(z) \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}. \tag{20}$$

In particular, we can reduce the general Papperitz equation to a much simpler form by using both transformations: we can make two of the exponents zero with (19), and move the singular points to (where else?) $0, 1, \infty$ using a Möbius transformation: we find that

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^\gamma P\begin{Bmatrix} a & b & c & \\ 0 & \beta+\alpha+\gamma & 0 & z \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix} \tag{21}$$

$$= \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^\gamma P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \beta+\alpha+\gamma & 0 & w \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix}, \tag{22}$$

where we have used the Möbius transformation

$$w = T(z) = \frac{(c-b)(z-a)}{(c-a)(z-b)}. \tag{23}$$

The solutions to the differential equation that this represents are extremely significant in their own right; we discuss them in the next section.

2 The Hypergeometric Function

It is traditional to start discussion of the hypergeometric function in quite a different place from what we were doing in the last section.

Definition 1. The *hypergeometric function* with parameters a, b and c is given by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1, \tag{24}$$

where $(a)_n$ is the *Pochhammer symbol* or *rising factorial*,

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{25}$$

when the right-hand side is defined.[5]

[5]For nonnegative integer n (which are all we consider), the Pochhammer symbol is really an entire function of a, which is rather obscured by the right-hand side, although it is normally easier to use for calculation. Also of note, naturally we have $(1)_n = n!$.

Obviously we have

$$F(a, b; c; z) = F(b, a; c; z). \tag{26}$$

There are various other notations used for this, including ${}_2F_1(a, b, c, z)$ and $F\left({a, b \atop c}; z\right)$, essentially because there are even more general generalisations obtained by adding more Pochhammer symbols, but somehow the 2-up-1-down version is the right amount of complication to be interesting, but easily tractable.

You should think of this as a generalisation of the function $(1-z)^a$, essentially because

$$F(a, b; b; z) = (1-z)^{-a}, \tag{27}$$

as can seen by direct calculation.[6] Actually, its far worse than that: F is actually a generalisation of loads of the functions you know and love: for example,

$$\begin{align} F(1, 1; 2; z) &= -\frac{\log(1-z)}{z}, \tag{28}\\ F(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; z^2) &= \frac{\arcsin z}{z} \tag{29}\\ F(1, \tfrac{1}{2}; \tfrac{3}{2}; -z^2) &= \frac{\arctan z}{z} \tag{30}\\ F(-a, a; \tfrac{1}{2}; \sin^2 z) &= \cos 2az \tag{31}\\ F(a, a+\tfrac{1}{2}; \tfrac{1}{2}; z^2) &= \frac{1}{2}\left((1+z)^{-2a} + (1-z)^{-2a}\right), \tag{32} \end{align}$$

and thousands more: almost any ordinary function you've written down over the years is one of these! Most of the above examples can be written down almost straight from the power series.[7]

One more thing: the radius of convergence of this series is strictly 1, but our numerous examples lead us to expect there is a natural analytic continuation of F. This is correct, and we will explore that shortly.

2.1 Integral representations and identities

If you've been paying attention, you may have noticed something about the coefficients in the power series of F:

$$\begin{aligned} \frac{(a)_n(b)_n}{(c)_n(1)_n} &= (-1)^n \binom{-a}{n} \frac{\Gamma(b+n)}{\Gamma(c+n)} \frac{\Gamma(c)}{\Gamma(b)} \\ &= (-1)^n \binom{-a}{n} \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \\ &= (-1)^n \binom{-a}{n} \frac{B(b+n, c-b)}{B(b, c-b)}. \end{aligned}$$

But we have an integral for the B-function:

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt \quad (\Re(z, w) > 0). \tag{33}$$

[6] Hence the rather mysterious and exciting name *hypergeometric*: it is a (vast) generalisation of a geometric series.

[7] Hopefully this has persuaded you that this is an object worthy of study in its own right.

Then we have

$$F(a,b;c;z) = \sum_{n=0}^{\infty}(-1)^n \binom{-a}{n} z^n \frac{1}{B(b,c-b)} \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt$$
$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \left(\sum_{n=0}^{\infty} (-1)^n \binom{-a}{n} (zt)^n \right) dt,$$

interchanging the order of summation and integration (fine since we have absolute convergence inside the radius of convergence). Hence

$$F(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt, \quad (\Re(b) > 0, \Re(c-b) > 0, |z| < 1). \tag{34}$$

Remark 2. Sadly this formula loses the symmetry between a and b, but we can recover it quite easily: Erdélyi had the bright idea that we can write the last bracket in the integral as

$$(1-zt)^{-a} = F(a,c;c;zt) = \frac{1}{B(a,c-a)} \int_0^1 s^{a-1}(1-s)^{c-a-1}(1-zst)^{-c}\,ds, \tag{35}$$

so we can write (34) as a double integral:

$$F(a,b;c;z) = \frac{1}{B(a,c-a)B(b,c-b)} \int_0^1 \int_0^1 \frac{s^{a-1}(1-s)^{c-a-1}t^{b-1}(1-t)^{c-b-1}}{(1-zst)^c}\,ds\,dt, \tag{36}$$

which is clearly symmetric in a and b. However, this integral is often too complicated to actually be useful.

Remark 3. The above integral only works for values of $\Re(b)$ and $\Re(c-b)$ such that the Beta integral converges. We can find a way around this, by using the contour reformulation of the Beta integral, taking the same integrand around the *Pochhammer contour*,[8] which encircles each singularity once clockwise and once anticlockwise. This defines F for any values of the parameters.

We can use (34) to produce several hypergeometric identities which were not apparent from the original series.

Firstly, putting $z = 1$ gives

$$F(a,b;c;1) = \frac{B(b,c-b-a)}{B(b,c-b)} = \frac{\Gamma(c-b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b) > 0, \tag{37}$$

which is Gauss's formula;[9] it is actually a generalisation of things like

$$\sum_{k=0}^{r} \binom{s}{k}\binom{t}{r-k} = \binom{s+t}{r}, \tag{38}$$

which is called the *Chu–Vandermonde identity*. (The derivation seems to require $\Re(c) > \Re(b) > 0$, but in fact, since both sides remain finite for $\Re(b) \leqslant 0$, there is no need to restrict (37) to b having positive real part: the important restriction is $\Re(c-a-b) > 0$.)

[8] Apparently due to Jordan, *Cours d'Analyse*, III, (1887).

[9] The vast majority of this section is due to Gauss, one way or another.

Secondly, if we change variables to $s = 1 - t$, the integral (34) becomes

$$F(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 (1-s)^{b-1} s^{c-b-1} (1-z(1-s))^{-a} \, ds \tag{39}$$

$$= (1-z)^{-a} \frac{1}{B(b,c-b)} \int_0^1 (1-s)^{b-1} s^{c-b-1} \left(1 - \frac{z}{z-1} s\right)^{-a} \, ds \tag{40}$$

$$= (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right), \tag{41}$$

and this gives us two things: one, a shiny new identity (which must by true for all values of a, b, c so that one side of this equation make sense, by analytic continuation), and two, if $|z/(z-1)| < 1$, z can live in the whole region $\Im(z) > 1/2$, as can be found by the usual conformal mapping calculation. This gives us a considerable analytic continuation of F as a function of z. This is called *Pfaff's transformation*.

We can obviously apply the same transformation to a and c, which gives us a set of three identities:

$$F(a,b;c;z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right) \tag{42}$$

$$= (1-z)^{-b} F\left(c-a, b; c; \frac{z}{z-1}\right) \tag{43}$$

$$= (1-z)^{c-a-b} F(c-a, c-b; c; z), \tag{44}$$

and the last of these is called *Euler's transformation*.

There are many other identities for hypergeometric functions: Gauss found a continued fraction representation, as well as many other relations between functions like $F(a,b;c;z)$ and $F(a+1,b;c;z)$: these last are called *contiguous relations*, and can often be used to reduce to simpler hypergeometric functions. However, we are interested in F because it satisfies a particular differential equation.

2.2 The hypergeometric differential equation

Consider the action of the operator $z\frac{d}{dz}$ on the hypergeometric series. We have

$$z\frac{d}{dz} F(a,b;c;z) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} n \frac{z^n}{n!}. \tag{45}$$

Combining several of these, we find that

$$\begin{aligned}
\left(z\frac{d}{dz} + a\right)\left(z\frac{d}{dz} + b\right) F &= \sum_{n=0}^{\infty} (n+a)(n+b) \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b_{n+1})}{(c)_{n+1}} \frac{z^{n+1}}{(n+1)!} \frac{(n+1)(c+n)}{z} \\
&= \frac{1}{z} \sum_{n=1}^{\infty} n(c+n-1) \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \\
&= \frac{1}{z} \sum_{n=0}^{\infty} n(c+n-1) \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \\
&= \frac{1}{z} \frac{d}{dz} \left(z\frac{d}{dz} + c - 1\right) F(a,b;c;z),
\end{aligned}$$

and so $u(z) = F(a, b; c; z)$ is a solution to the differential equation

$$z\left(z\frac{d}{dz} + a\right)\left(z\frac{d}{dz} + b\right)u - z\frac{d}{dz}\left(z\frac{d}{dz} + c - 1\right)u = 0, \tag{46}$$

which, after multiplying out, can be written suggestively as

$$z(1-z)u'' + (c - (a+b+1)z)u' - abu = 0, \tag{47}$$

which is called the *hypergeometric differential equation.* Dividing through gives

$$u'' + \left(\frac{c}{z} + \frac{a+b-c+1}{z-1}\right)u' + \frac{ab}{z(z-1)}u = 0. \tag{48}$$

We recognise this as a Papperitz equation, with singular points at 0, 1 and ∞. We would like to know what the exponents are. Perhaps the simplest way to do this is to deduce the form of the Papperitz equation when the singular points are at $0, 1, \infty$:

$$u'' + \left(\frac{1-\alpha-\alpha'}{z} + \frac{1-\beta-\beta'}{z-1}\right)u' + \frac{1}{z(z-1)}\left(-\frac{\alpha\alpha'}{z} + \frac{\beta\beta'}{z-1} + \gamma\gamma'\right)u = 0, \tag{49}$$

the other terms disappearing in the limit. It is immediately apparent that in the hypergeometric case, we can take $\alpha = 0$, so $\alpha' = 1-c$; proceeding in this way, we obtain the P-symbol for the hypergeometric equation (47):

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix}. \tag{50}$$

Now, digging into the back of your mind should be the question: *what's the other solution of* (47)*?* We can find a linearly independent solution at 0 using the P-symbol:

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix} = z^{1-c}P\begin{Bmatrix} 0 & \infty & 1 & \\ c-1 & a-c+1 & 0 & z \\ 0 & b-c+1 & c-a-b & \end{Bmatrix}, \tag{51}$$

which, when we swap the exponents for 0 (the ordering being irrelevent), is again in the form of a P-symbol for a hypergeometric equation: specifically, one satisfied by $F(a-c+1, b-c+1; 2-c; z)$. Therefore the two solutions at $z = 0$ are $F(a, b; c; z)$ and $z^{1-c}F(a-c+1, b-c+1; 2-c; z)$: we can tell they are linearly independent because they have different exponents.

2.3 Solution of the Papperitz equation using hypergeometric functions

The relationship between the hypergeometric function and the Papperitz equation is two-way, however: recall that at the end of § 1.3, we reduced the form of the P-symbol for a general Papperitz equation to that for the hypergeometric equation:

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma}P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \beta+\alpha+\gamma & 0 & \frac{(c-b)(z-a)}{(c-a)(z-b)} \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix}. \tag{52}$$

We can identify the parameters: the easiest way is to read off a and b from the ∞ column, and then $1-c$ from the 0 column. Hence the general Papperitz equation has hypergeometric solution

$$u = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma}F\left(\alpha+\beta+\gamma, \beta'+\alpha+\gamma; 1+\alpha-\alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)}\right). \tag{53}$$

(We shall assume throughout this section that none of the exponent differences $\alpha - \alpha'$, $\beta - \beta'$ or $\gamma - \gamma'$ is an integer: this simplifies the theory, since otherwise we'll have logarithmic solutions, just as in typical method of Frobenius.)

But we made a choice in this derivation: we treated α and α' differently. The derivation goes through in exactly the same way if we swap α and α', or γ and γ', or both pairs, since the Papperitz equation is unchanged. Hence we obtain three more solutions. Even worse, the equation is indifferent to the ordering of the columns in the P-symbol, so we could choose a different ordering of a, b and c to apply this scheme to. It is apparent that all of these orderings give different results in the final P-symbols, so there are in fact $3! \times 4 = 24$ different hypergeometric solutions to the Papperitz equation![10]

What is the relationship between all these? They solve a second-order linear differential equation, so between any three there must be a linear relationship. For solutions with the same ordering of a, b, c, it is easy to determine this using the power series, but when the points are in a different order, it is *much* harder.

2.4 Barnes's integral

Barnes happened upon a very natural way to describe the hypergeometric function, based on the idea of a Mellin transform.

Definition 4 (Mellin transform). Let $f(x)$ be an function on $(0, \infty)$, bounded by x^k for some $k > 0$. The *Mellin transform* of f is

$$M(f)(s) = \int_0^\infty x^{s-1} f(x)\, dx, \quad 0 < \Im(s) < k. \tag{54}$$

Now, don't make that "Oh no, not another transform" expression: this one's not for solving differential equations: it instead has uses in function theory.[11] Notice that the transformed function is in general an analytic function on the strip $0 < \Im(s) < k$; this leaves us with little choice for how to invert it, since a continuation of $M(f)$ using the usual monodromy methods could (and in general, will) produce poles on both sides of the strip. Hence the *Mellin inversion formula* is really the only possible thing it could be:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} M(f)(s)\, ds, \tag{55}$$

where $0 < c < k$, and the contour lies (at least initially) entirely in the strip $0 < \Im(s) < k$. This can be proven using the Fourier inversion theorem.

Actually computing the inversion tends to require continuation of $M(f)$ out of the strip, and wrapping the contour around the poles, much like the Laplace transform.

This might ring some bells in the back of your mind: if I'm picking up contributions from poles, the residues will look like $x^{-a} \operatorname{Res}_{s=a} M(f)(s)$, at least for simple poles. But this looks like a power series expansion!

Let's look at an example. A really simple function on the positive real axis is $f(x) = e^{-\alpha x}$. We find

$$M(f)(s) = \int_0^\infty x^{s-1} e^{-\alpha x}\, dx = \frac{\Gamma(s)}{\alpha^s}. \tag{56}$$

[10] This was first shown by Kummer. I shall spare you the full list: it is available in numerous places, and we are very fortunate that the notation for this subject is unusually consistent, so at least all my sources use the same notation.

[11] It also has extensive uses in number theory, but you can look those up yourself.

We can now apply the Mellin inversion formula: we want to calculate

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{\alpha^s}x^{-s}\,ds.really \tag{57}$$

Ah, but the Γ-function has simple poles at $s = -n$, $n = 0, 1, 2, \ldots$, with residues $(-1)^n/n!$. We can also show (e.g. using the product expansion) that $\Gamma(s) \to 0$ exponentially as $\Re(s) \to -\infty$ if s is away from the real line, so all we have to do is add up the residues. We then find that the residues are all of the form $(-1)^n/(n!\alpha^{-n}x^{-n})$, and so

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{\alpha^s x^s}\,ds = \sum_{n=0}^{\infty}\frac{(-1)^n\alpha^n x^n}{n!}, \tag{58}$$

which matches the power series we know and love for the negative exponential!

I sense you are unimpressed. Wake up, this is the good bit! We can go the other way, from the power series to the integral! In particular, we can do this for the hypergeometric series: start with a vertical contour, and expect to complete it to the left. Then $(-z)^s\Gamma(-s)$ has residue $-z^n/n!$ at each nonnegative integer n, and if we complete the contour to the left, the sense is negative, so these will become positive. To get the other factors in the power series, remember that the coefficient contains $\Gamma(a+n)\Gamma(b+n)/\Gamma(c+n)$; there's a really obvious analytic continuation of this function from n to general complex s. There is therefore really only one possible thing to write down:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s\,ds, \tag{59}$$

where the contour must pass to the *left* of the poles of $\Gamma(-s)$, but to the *right* of the poles of the Γs containing a and b, so we don't pick up the extra poles from them.[12] (Obviously this requires that a, b are not negative integers, in which case the hypergeometric function is just a polynomial anyway.) This integral is called *Barnes's integral*, or sometimes the *Mellin-Barnes integral*. One can apply the reflection formula $\Gamma(-s)\Gamma(1+s) = -\pi\operatorname{cosec}\pi s$ and a variant of Stirling's formula to show that the Γ factors essentially cancel out, and the integrand decays exponentially provided that $|\arg z| < \pi - \delta$.[13] Therefore the integral defines an analytic function of z on the plane without the negative real axis.

Now, we should check that this formula gives the expression we want. A calculation using Stirling shows that on the semicircle C on the right of the imaginary axis of radius $N+1/2$, N a positive integer, we have

$$\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)} = O(N^{a+b-c-1}) \tag{60}$$

as $N \to \infty$. Meanwhile, an elementary calculation shows that on the same circle, if $|\arg z| < \pi - \delta$, then

$$(-z)^s\operatorname{cosec}\pi s = O\left(\exp\left(\left(N+\tfrac{1}{2}\right)(\cos\theta\log|z| - \delta\,|\sin\theta|)\right)\right), \tag{61}$$

which shows that if $\log|z|$ is negative (so $|z| < 1$), the integrand tends to zero exponentially over the whole of the semicircle. Therefore, taking the integral around the semicircle C and the appropriate truncation of the original contour, we obtain via Cauchy's theorem

$$\frac{1}{2\pi i}\int_{-i(N+1/2)}^{i(N+1/2)}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s\,ds - \int_C = -2\pi i\sum_{n=0}^{N}\frac{1}{2\pi i}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{-z^n}{n!}, \tag{62}$$

[12] Yet….

[13] See e.g. Whittaker and Watson, p. 287 for the details of this

and letting $N \to \infty$ shows that provided $|\arg z| < \pi - \delta$ and $|z| < 1$, the original integral is equal to

$$\sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{-z^n}{n!} = F(a,b;c;z). \tag{63}$$

Therefore the integral provides an analytic continuation of the series for F to the complex plane minus the negative real axis. In some ways, this is "the best" definition of F that we have: it has easy generalisations to other hypergeometric functions,[14] and (as we discuss next) it can even be used to prove Kummer's connexion formulae.

2.5 Kummer's connexion formulae

In this section we discuss some of the relationships between the 24 solutions described in § 2.3. A preliminary investigation of the expansions, along with use of Euler's transformation, leads us to six distinct functions, two for each singular point:

$$\begin{aligned}
y_{01} &= F(a,b;c;z) && (64)\\
y_{0c} &= z^{1-c}F(a-c+1,b-c+1;2-c;z) && (65)\\
y_{11} &= F(a,b;a+b-c+1;1-z) && (66)\\
y_{1cab} &= (1-z)^{c-a-b}F(c-a,c-b;c-a-b+1;1-z) && (67)\\
y_{\infty b} &= (-z)^{-b}F(a,a-c+1;a-b+1;1/z) && (68)\\
y_{\infty a} &= (-z)^{-a}F(b,b-c+1;b-a+1;1/z); && (69)
\end{aligned}$$

that's the easy bit. We shall now obtain relationships between y_{00} and the last two pairs of functions: since they all solve the hypergeometric equation, there must be a linear relation between them.

Theorem 5. *When $a-b$ is not an integer, we have the identity*

$$\begin{aligned}
F(a,b;c;z) =& \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a}F(a,a-c+1;a-b+1;1/z) && (70)\\
&+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b}F(b,b-c+1;b-a+1;1/z) && (71)
\end{aligned}$$

for $|\arg(-z)| < \pi$.

We shall prove this result in two ways: the first uses Barnes's integral with a different closure of the contour. The second is the way you'd probably think of yourself if left to your own devices, using Pfaff's transformation.

First Proof. Suppose that $|z| > 1$ and $|\arg(-z)| < \pi$, and close the contour for the Barnes integral using a semicircle D of radius ρ on the left of the imaginary axis. We can then show, by a similar argument to the previous section, that choosing an appropriate sequence of radii $\rho_N \to \infty$ allows the integral over the semicircle D to decay to zero: we take this limit. Cauchy's theorem now says that the Barnes integral is equal to $2\pi i$ times the sum of the residues at the poles of $\Gamma(a+s)$ and $\Gamma(b+s)$.

By the symmetry, we only need to do one of the sums, and the other can be obtained by interchanging the rôles of a and b. $\Gamma(a+s)$ has simple poles at $s = -a-n$, and the residues are then

$$\frac{1}{2\pi i}\frac{(-1)^n}{n!}\frac{\Gamma(b-a-n)\Gamma(a+n)}{\Gamma(c-a-n)}(-z)^{-a-n} = (-z)^{-a}\frac{\Gamma(a+n)\Gamma(a-c+1+n)}{\Gamma(a-b+1+n)}\frac{\pi\sin(c-a-n)\pi}{\pi\sin(b-a-n)\pi}\frac{z^{-n}}{n!}, \tag{72}$$

[14] And look up the Meijer G-function and Fox H-function if you would like your mind further boggled.

and summing this over the nonnegative integers gives

$$\begin{aligned}&\frac{\pi\sin{(c-a)\pi}}{\pi\sin{(b-a)\pi}}(-z)^{-a}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(a-c+1+n)}{\Gamma(a-b+1+n)}\frac{z^{-n}}{n!}\\&=\frac{\pi\sin{(c-a)\pi}}{\pi\sin{(b-a)\pi}}(-z)^{-a}\frac{\Gamma(a)\Gamma(a-c+1)}{\Gamma(a-b+1)}F(a,a-c+1;a-b+1;1/z)\end{aligned}\tag{73}$$

and so we obtain, after using the reflection formula again and including the other series,

$$\begin{aligned}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;z)=&\frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a}F(a,a-c+1;a-b+1;1/z)\\&+\frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b}F(b,b-c+1;b-a+1;1/z),\end{aligned}\tag{74}$$

and dividing through gives the result. □

This is quite unintuitive, and contains some heavy algebraic and analytic lifting. In the second proof, we get around this by sacrificing some initial generality.

Second proof. We know the relation must take the form

$$\begin{aligned}F(a,b;c;z)=&A(-z)^{-a}F(a,a-c+1;a-b+1;1/z)\\&+B(-z)^{-b}F(b,b-c+1;b-a+1;1/z),\end{aligned}\tag{75}$$

for some constants A and B. We can initially assume that $\Re(b)>\Re(a)$, and then as $|z|\to\infty$, the right-hand side of this equation is asymptotically $A(-z)^{-a}$. On the other hand, one of Pfaff's transformations gives

$$F(a,b;c;z)=(1-z)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right)\sim(1-z)^{-a}F(a,c-b;c;1),\tag{76}$$

and Gauss's formula (37) tells us the value of this when $\Re(c-a-(c-b))=\Re(b-a)>0$, which is the restriction we imposed. Hence we obtain

$$A=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}.\tag{77}$$

Now, interchanging a and b in of (75) leaves the left-hand side unchanged, while interchanging a and b on the right is equivalent to interchanging A and B. Hence we must have

$$B=\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)},\tag{78}$$

and we conclude that

$$\begin{aligned}F(a,b;c;z)=&\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a}F(a,a-c+1;a-b+1;1/z)\\&+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b}F(b,b-c+1;b-a+1;1/z).\end{aligned}\tag{79}$$

But since this is symmetric in a and b, the restriction $\Re(b)>\Re(a)$ is not actually required in this formula, and we are done. □

We can carry out a similar argument for y_{00}, y_{10} and y_{11}, although it is slightly more complicated in detail.

Theorem 6. *Suppose that $c-a-b$ is not an integer. Then for $|\arg(1-z)| < \pi$,*

$$\begin{aligned} F(a,b;c;z) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} F(c-a,c-b;c-a-b+1;1-z). \end{aligned} \tag{80}$$

We can also do this two ways, but we leave the Barnes integral method to the reader: it requires the integral representation

$$\frac{\Gamma(a)}{(1-z)^a} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a+s)\Gamma(-s)(-z)^s \, ds, \tag{81}$$

which is a simple corollary of the general Barnes integral, and Barnes's lemma

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)\, ds = \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}, \tag{82}$$

where the poles of $\Gamma(\gamma-s)\Gamma(\delta-s)$ lie on the right of the contour, and those of $\Gamma(\alpha+s)\Gamma(\beta+s)$ on the left. To avoid this, we shall carry out a similar proof to our second proof of the previous theorem.

Proof. Since the three hypergeometric functions in the theorem are all solutions to the same differential equation on a common domain, there must be a relation of the form

$$\begin{aligned} F(a,b;c;z) =& AF(a,b;a+b-c+1;1-z) \\ &+ B(1-z)^{c-a-b} F(c-a,c-b;c-a-b+1;1-z). \end{aligned} \tag{83}$$

between them. Suppose first that $\Re(c-a-b) > 0$. Then we can put $z=1$ in this equation, which gives immediately

$$A = F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{84}$$

once again by Gauss's formula (37). Now, if we assume that $\Re(1-c) > 0$, we can set $z=0$ to obtain

$$1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} 1 + BF(c-a,c-b;c-a-b+1;1), \tag{85}$$

and after another application of Gauss's formula, and some fairly painful reflection formula work, we eventually find that

$$B = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \tag{86}$$

To remove the restrictions on $c-a-b$ and $1-c$, we apply the Euler transformation

$$F(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta;\gamma;z) \tag{87}$$

to the left-hand side of (83), the right-hand side of (83), or both at once: a calculation shows that in all cases, A and B have the same values. Hence the relationship holds, with the same constants, for all nonzero values of $\Re(1-c)$ and $\Re(c-a-b)$; we can then use that both sides are meromorphic functions of a, b, c for c and $c-a-b$ not integers to continue over the remaining lines of zero real part, excluding $c=0$ and $c-a-b=0$, of course. Hence we have the result. □

3 Monodromy

Differential equations discuss local behaviour of functions, but their solutions are somehow meant to be global objects: when the differential equation has regular singular points, solutions to the differential equation in general have branch points at the regular singular points, and hence are not quite global objects on the Riemann sphere. To fix ideas, suppose we have an nth-order differential equation with only m regular singular points on $\hat{\mathbb{C}}$.

So there comes the obvious question: what happens when we walk once around a regular singular point at $z = a$? We must end up with a solution to the differential equation, but because the functions change branch, it may not be the same one we started with. On the other hand, since both the new and the old solutions are valid on a common domain, the new solutions must be a linear combination of the old ones. Therefore, if we have old solutions y_i and new solutions Y_i, we must have

$$y_i = \sum_{i=1}^{n} M_{ij}^{(a)} y_j \tag{88}$$

for some fixed $n \times n$ matrix $M^{(a)}$. These are called *monodromy matrices*. If we encircle a anticlockwise again, we multiply by $M^{(a)}$ again, and so on; if we go around another point b in the region of validity of these solutions, we multiply by another matrix, $M^{(b)}$, an so on. The group generated by the products of all the matrices is called the *monodromy group*.[15]

As a first example, consider the Euler–Cauchy equation on $\hat{\mathbb{C}} \setminus \{0, \infty\}$:

$$y'' + \frac{1 - \alpha + \alpha'}{z} y' + \frac{\alpha\alpha'}{z^2} y = 0; \tag{89}$$

this has the Riemann P-symbol

$$P \begin{Bmatrix} 0 & \infty & \\ \alpha & -\alpha & z \\ \alpha' & -\alpha' & \end{Bmatrix}, \tag{90}$$

and as we know, solutions with exponents α and α' are given by

$$y_\alpha = z^\alpha \qquad y_{\alpha'} = z^{\alpha'}. \tag{91}$$

If we encircle the origin once anticlockwise, we obtain a different set of solutions, but these are given only by constant multiples:

$$Y_\alpha = (e^{2\pi i} z)^\alpha = e^{2\pi i \alpha} y_\alpha \tag{92}$$

and similarly for $Y_{\alpha'}$, so the matrix becomes

$$M^{(0)} = \begin{pmatrix} e^{2\pi i \alpha} & 0 \\ 0 & e^{2\pi i \alpha'} \end{pmatrix}. \tag{93}$$

If one goes anticlockwise around ∞, it is equivalent to going clockwise around the origin, and hence $A^\infty = (A^{(0)})^{-1}$. Therefore the group of matrices is generated by $M^{(0)}$; if $\alpha \neq \alpha'$ are both rational, the group is finite, but if one is irrational, the group is infinite.

Notice also that the matrix depends on the basis of solutions: it is only nice and diagonal because we chose the "right" basis of solutions around 0; in general, matrices will not be diagonal.

[15] This topic appears to have been first studied by Riemann, in an unpublished paper.

3.1 The fundamental group

Let's think about what we're actually doing for a second. We have loops on a space, and a group of matrices. What about the loops themselves? Let the space we are working in (currently a punctured surface) be X The paths that can be drawn starting and ending at a point P on a surface form a group.[16] It's a very big group, however, and not very intuitive, so we quotient it by the equivalence relation that two paths γ, δ are equivalent if $\gamma \circ \delta^{-1}$ is continuously deformable (*homotopic*) to a point. This new group is called the *fundamental group* of the surface X, and is denoted $\pi_1(X)$. Now, the relevence of this, of course, is that we are considering loops on the surface of the punctured sphere, $X = \hat{\mathbb{C}} \setminus \{a_1, \dots, a_n\}$, and we have discovered that to each loop, we can associate a matrix. A moment's consideration shows that the map $\pi_1(X) \to GL(n)$ is a group homomorphism, i.e., if I compose two loops, the total action on my function is given by composing the matrices. Therefore the monodromy group is a *representation* of $\pi_1(X)$.

Our first example was rather a dull one, because the fundamental group of the doubly-punctured sphere is abelian.[17] The first interesting example is when we have a sphere with three points removed, as in the Papperitz equation, which we'll discuss briefly below.

The general case is very complicated. However, it is possible (and you'll have to take my word for this) in general to order the a_i such that if we arrange loops γ_i around each a_i, we have $\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_n = 1$ (think about cutting the loops at the crossovers, like in contour integration). Hence we must have

$$M^{(a_1)} M^{(a_2)} \cdots M^{(a_m)} = I. \tag{94}$$

(And in fact, this is the only possible condition we can have on the matrices.)

In general, one can go from the differential equation to its monodromy group, although it's not very easy. The inverse problem, which for some reason is another thing called the Riemann–Hilbert problem, is much harder: given any group of matrices with a relation (94), it *is* possible to find a differential equation with the group generated by $M^{(a_i)}$ as its monodomy group.

3.2 Monodromy of the Papperitz equation

Okay, now let's do a very simple example for the Papperitz equation, to get our feet back on the ground. Here, it is possible that we have a nonabelian group, because we have three matrices, $M^{(a)}, M^{(b)}, M^{(c)}$, which may not commute. We can specialise to the case $(a, b, c) = (0, 1, \infty)$, and then it is clear that the group is generated by $M^{(0)}$ and $M^{(1)}$, since either $M^{(0)} M^{(1)} M^{(\infty)} = I$ or some other permutation.

We can look at a simple example. Let our basis be $y_{00} = F(a, b; c; z)$ and $y_{0c} = z^{1-c} F(\dots)$, (in the notation of § 2.5 the other solution at 0. Hence we have, with respect to this basis y_i, the matrix

$$M^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i c} \end{pmatrix}, \tag{95}$$

cancelling the redundant $e^{2\pi i}$. Obviously this is nice because we chose the "right" basis. Finding the monodromy matrix at 1 is much worse. We can operate by a change of basis. A tedious computation like the ones for Kummer's connexion formulae shows that

$$\mathbf{y}^{(1)} = \begin{pmatrix} y_{11} \\ y_{1cab} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_{01} \\ y_{0c} \end{pmatrix} = S\mathbf{y}^{(0)}, \tag{96}$$

[16] proof: associativity is either clear or awful, depending on how careful you are. If γ, δ are admissible paths, then so is $\gamma \circ \delta^{-1}$.

[17] And in particular, $\pi_1(\hat{\mathbb{C}} \setminus \{0, \infty\}) \cong \mathbb{Z}$. You can easily check that a parallel of our argument for the matrix group shows that the only (normal) subgroups of $\mathbb{Z}$ are the finite or infinite cyclic groups, in agreement with the possible representations we found.

say, where A, B, C, D are all products of four Γ-functions—you can see `http://dlmf.nist.gov/15.10#E17` for the gory details (we know the matrix S is nonsingular since the functions on each side form a basis). Now, we know how y_{10} and y_{1cab} transform at $z = 1$: with respect to this basis, the monodromy matrix is

$$M'^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(c-a-b)} \end{pmatrix}. \tag{97}$$

To return to the original basis $\mathbf{y}^{(0)}$, we have to undo the transformation to $\mathbf{y}^{(1)}$, so of course

$$\mathbf{Y}^{(0)} = M^{(1)}\mathbf{y}^{(0)} = S^{-1}\mathbf{Y}^{(1)} = S^{-1}M'^{(1)}S\mathbf{y}^{(0)}, \tag{98}$$

so $M^{(1)} = S^{-1}M'^{(1)}S$, and it will be a brave mathematician who calculates it explicitly ….

The above was a very simple example of a monodromy calculation: in general, it's much worse, because you don't have the connexion formulae initially to help you out!

References

There are a lot of nice references for these classic topics in special functions: you might consider

- Whittaker and Watson, *A Course in Modern Analysis* (CUP, 1927).
  § 10.7 for the Riemann–Papperitz theory and Chap. 14 for the hypergeometric function.
- Kohno, M., *Global Analysis in Linear Differential Equations* (Springer, 1999).
  Ch. 2 covers the hypergeometric function, and § 2.2 has a thorough discussion of the connexion formulae.
- Wang, Z. X. and Guo, D. R., *Special Functions* (World Scientific, 1989).
  § 4.8 has the nice proof of the connexion formulae that we prefer.
- Beukers, F. *Notes on differential equations and hypergeometric functions* (`http://pages.uoregon.edu/njp/beukers.pdf`, retrieved March 2016)
  A set of notes on monodromy and how it works for the hypergeometric function in much more detail than we had space to discuss.
- Riemann, B. *Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Functionen.*, in Aus dem siebenten Band der Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1857. Reprinted in *Gesammelte mathematische Werke* (Leipzig, 1876), Chapter IV, pp. 62–78.
  Riemann's original paper on the P-symbol, discussed without any calculation of the differential equation. Riemann evidently deducted its existence from the hypergeometric equation, but this lack of algebraic calculation is absolutely typical of Riemann's mathematical style.
- Riemann, B. *Zwei allgemeine Sätze über lineäre Differentialgleichungen mit algebraischen Coefficienten*, in *Gesammelte mathematische Werke* (Leipzig, 1876), Chapter XXI, pp. 357–369.
  Riemann's original paper on the monodromy of linear differential equations with algebraic coefficients.

Kummer's 24 solutions to the Papperitz equation can be found in

- Whittaker and Watson, *A Course in Modern Analysis*, § 14.3, pp. 283–285,
- Abramowitz and Stegun, *Handbook of Mathematical Functions*, § 15.5, particularly formulae 15.5.3–15.5.14, pp. 562–3.
- Digital Library of Mathematical Functions, § 15.10, `http://dlmf.nist.gov/15.10`.