Selected answers to Further Complex Methods Example Sheet 2: Special Functions

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1 Question 4: Multiplication formula for the Γ-function

We wish to show that

$$\prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right) = m^{1/2-mz}(2\pi)^{(m-1)/2}\Gamma(mz) \tag{1}$$

The question obviously hints that we can prove this as a generalisation of the $m = 2$ case, using the Weierstrass product definition. Let's do that first.

1.1 With the Weierstrass product

Let's start with the Weierstrass product for the Γ-function, which is

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{1}{1+z/n} e^{z/n}. \tag{2}$$

We consider the function

$$g(z) := \frac{m^{mz-1/2}\Gamma(z)\Gamma(z+1/m)\cdots\Gamma(z+(m-1)/m))}{\Gamma(mz)} = \frac{m^{mz-1/2}}{\Gamma(mz)} \prod_{k=0}^{m-1} \Gamma\left(z+\frac{k}{m}\right), \tag{3}$$

and expand the Γ-functions using the product formula:

$$\begin{aligned} g(z) =& m^{mz-1/2} e^{-\gamma(mz+(m-1)/2-mz)} \frac{mz(1+mz)\cdots(1+mz/(m-1))}{z(z+1/m)\cdots(z+(m-1)/m)} e^{m(z/1+z/2+\cdots+z/(m-1))} \\ & \times \prod_{n=1}^{\infty}\prod_{k=0}^{m-1} \frac{1+mz/(mn+k)}{1+(z+k/m)/n} e^{(z+k/m)/n - mz/(mn+k)}, \end{aligned} \tag{4}$$

where we have used the important fact that every positive integer N can be written as

$$N = mn + k, \qquad n \geqslant 1, \quad 0 \leqslant k \leqslant m-1 \tag{5}$$

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to split the $\Gamma(mz)$ product into factors that can cancel with factors in the other Γ products: this is the key insight, and everything from here is just nasty algebra. The mess outside the product is required since the initial factors are of a different form. The first thing to do is cancel the fractions, i.e., calculate

$$\frac{1+mz/(mn+k)}{1+(z+k/m)/n} = \frac{mn}{mn+k}\frac{mn+k+mz}{nm+mz+k} = \frac{mn}{mn+k}, \tag{6}$$

so they all cancel into constants. Similarly, the exponential has argument

$$-\frac{mz}{mn+k} + \frac{mz+k}{mn} = \frac{k}{mn} - \frac{mz}{(mn+k)} + \frac{z}{n}. \tag{7}$$

This will be easy to understand if written in the right way: the first term is constant, so we can ignore it. The second term produces an absolutely convergent series, so we can peel it off the main product. We are down to

$$g(z) = m^{mz-1/2}e^{-(m-1)\gamma/2}\frac{m^m}{1\cdot 2\cdots(m-1)}\left(\prod_{n=1}^{\infty}\prod_{k=0}^{m-1}\frac{mn}{mn+k}e^{k/mn}\right)\exp\left(\sum_{n=0}^{\infty}{}'\sum_{k=0}^{m-1}\frac{1}{mn}-\frac{1}{mn+k}\right)mz, \tag{8}$$

where the primed sum leaves out any terms with zero denominator. Now, there is only one possible thing that can happen here in order for the zs to cancel: we need the double sum to evaluate to $-\log m$. To see this, notice that if we write the sum slightly differently, we can evaluate the finite sum as

$$\sum_{n=0}^{N}{}'\left(\sum_{k=1}^{m}\frac{1}{mn+k}-\frac{1}{mn}\right) = H_{m(N+1)} - H_N, \tag{9}$$

where $H_r = \sum_{k=1}^{r} 1/k$ is the rth *Harmonic number*. We know that $H_r = \log r + \gamma + o(1/r)$ as $r \to \infty$ by a standard integral test argument, so as $N \to \infty$,

$$H_{m(N+1)} - H_N = \log m(N+1) + \gamma - \log N - \gamma + o(1/N) = \log m + o(1/N), \tag{10}$$

and hence we conclude that the sum is equal to $-\log m$ as we supposed. We are left with

$$g(z) = \frac{m^{m-1/2}e^{(m-1)\gamma/2}}{(m-1)!}\prod_{n=1}^{\infty}\prod_{k=0}^{m-1}\frac{1}{1+k/mn}e^{k/mn}, \tag{11}$$

which we thankfully don't have to massage any more since it's all constant.

What is the constant? Setting $z = 1/m$ gives

$$g(z) = m^{1/2}\Gamma\left(\frac{1}{m}\right)\dots\Gamma\left(\frac{m-1}{m}\right)\frac{\Gamma(1)}{\Gamma(1)}. \tag{12}$$

We can cheat here and use the reflection formula,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{13}$$

we know that this constant is positive, and so if we square, we can pair off k/m and $1-k/m$ factors to obtain

$$g(z)^2 = \frac{m\pi^{m-1}}{\prod_{k=1}^{m-1}\sin\left(\frac{k\pi}{m}\right)}. \tag{14}$$

You may well know the answer to this: we have

$$\frac{\sin nz}{\sin z} = 2^{n-1} \prod_{k=1}^{n-1} \sin\left(z + \frac{k\pi}{n}\right), \tag{15}$$

(there's a nice proof of this using the multiple-angle formulae and some polynomial tricks, which I may write up separately at some point), and taking $z \to 0$ will give the constant we desire; taking a square root gives the constant as $(2\pi)^{(m-1)/2}$, as required.

There is, however, another way of doing this, which is somewhat less ugly.

1.2 With Stirling's formula

On the other hand, if we have Stirling's asymptotic formula,

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z}, \tag{16}$$

many of the relations for the Γ-function can be computed using a cute trick. We again examine the ratio

$$g(z) = \frac{m^{mz-1/2}\Gamma(z)\Gamma(z+1/m)\cdots\Gamma(z+(m-1)/m))}{\Gamma(mz)}, \tag{17}$$

but this time, we first show that it is periodic with period 1:

$$\frac{g(z+1)}{g(z)} = \frac{m^{m(z+1)-1/2}}{m^{mz-1/2}} \frac{\Gamma(mz)}{\Gamma(mz+m)} \prod_{k=0}^{m-1} \frac{\Gamma(z+1+k/m)}{\Gamma(z+k/m)} \tag{18}$$

$$= m^m \frac{1}{(mz+m-1)(mz+m-2)\cdots(mz+1)(mz)} \prod_{k=0}^{m-1} (z+k/m) = 1 \tag{19}$$

(this works if none of the factors is zero: we can deal with those cases by taking limits).

Now we can use Stirling's formula: as $z \to \infty$,

$$g(z) \sim \frac{m^{mz-1/2}}{\sqrt{2\pi} m^{mz-1/2} z^{mz-1/2} e^{-mz}} \prod_{k=0}^{m-1} \sqrt{2\pi} z^{z+k/m-1/2} e^{-z} \tag{20}$$

$$\sim (2\pi)^{(m-1)/2} z^{1/2-mz+mz+(m-1)/2-m/2} = (2\pi)^{(m-1)/2}. \tag{21}$$

But therefore, when n an integer, we have that for any z,

$$\lim_{n\to\infty} g(z+n) = (2\pi)^{(m-1)/2}; \tag{22}$$

since the only periodic functions with definite limits are constants, we have $g(z) = (2\pi)^{(m-1)/2}$ for any $z \in \mathbb{C}$.

Remark 1. You may be wondering about the terms I threw away without a mention, i.e., the $(1 + k/mz)^{z+k/m-1/2} e^{-k/m}$ in the sum. Obviously $(1+k/mz)^{k/m-1/2} \sim 1 + O(1/z)$, because the exponent is fixed. On the other hand, you should recognise that

$$(1+k/mz)^{z+k/m-1/2} \to e^{k/m} \tag{23}$$

as $z \to \infty$, and so in fact it was reasonable to discard these.

2 Question 10: An integral for the Riemann ζ-function

We have the integral

$$J = \int_\gamma \frac{\log t}{e^{-t}-1}\,dt, \tag{24}$$

where γ is the Hankel contour, and $\log$ has a branch cut along the negative real axis. We must split this into three sections, which I take to be the path just below the negative real axis γ_1, up to $t=-\varepsilon$, the circle γ_2, centred at $t=0$ of radius ε, and the path γ_3 just above the negative real axis, from $-\varepsilon$. Then we have $t = re^{-i\pi}$ and $\log t = \log r - i\pi$ on γ_1, so

$$\int_{\gamma_1} = \int_{-\infty}^{-\varepsilon} \frac{\log t}{e^{-t}-1}\,dt = -\int_\varepsilon^\infty \frac{\log r - i\pi}{e^r-1}\,dr, \tag{25}$$

while on γ_3, $t = re^{i\pi}$ and $\log t = \log r + i\pi$, so similarly

$$\int_{\gamma_3} = \int_\varepsilon^\infty \frac{\log r + i\pi}{e^r-1}\,dr, \tag{26}$$

and the log terms cancel, leaving us with

$$\int_{\gamma_1\cup\gamma_3} = 2\pi i \int_\varepsilon^\infty \frac{dr}{e^r-1}. \tag{27}$$

Now, we can actually do this integral explicitly:

$$\int_\varepsilon^\infty \frac{dr}{e^r-1} = \int_\varepsilon^\infty \frac{e^{-r}\,dr}{1-e^{-r}} = \left[-\log\left(1-e^{-r}\right)\right]_\varepsilon^\infty = -\log\left(1-e^{-\varepsilon}\right), \tag{28}$$

and from here, we find that

$$\int_{\gamma_1\cup\gamma_3} = 2\pi i \log\varepsilon + O(\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{29}$$

Right, now we have to deal with the integral over γ_2, where $t = \varepsilon e^{i\theta}$, so

$$\int_{\gamma_2} = \int_{-\pi}^{\pi} \frac{\log\varepsilon + i\theta}{\exp\left(-\varepsilon e^{i\theta}\right)-1}\, i\varepsilon e^{i\theta}\,d\theta. \tag{30}$$

Expanding the integrand as a series in ε, we find

$$\frac{\log\varepsilon + i\theta}{\exp\left(-\varepsilon e^{i\theta}\right)-1}\, i\varepsilon e^{i\theta} = (\log\varepsilon + i\theta)(-i) + O(\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{31}$$

Since θ is an odd function, its integral vanishes, and we are left with

$$\int_{\gamma_2} = -2\pi i \log\varepsilon + O(\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{32}$$

This cancels the contributions from the other parts of the contour, and we find $J=0$.

Right, now how on earth are we going to get to $\zeta(s) - 1/(s-1)$ from here? This integral is the derivative of the integral in Question 7, evaluated at $s=1$. That integral is given by

$$\frac{1}{2\pi i}\int_\gamma \frac{t^{s-1}}{e^{-t}-1}\,dt = \frac{\zeta(s)}{\Gamma(1-s)} = -\frac{(s-1)\zeta(s)}{\Gamma(2-s)}. \tag{33}$$

It is easy to check using the Residue Theorem that this integral has value -1 when $s = 1$, by deforming the contour to a small circle around the origin. Now, to compute the limit, we have

$$0 = J = \lim_{s \to 1} \frac{1}{s-1} \left(\frac{1}{2\pi i} \int_\gamma \frac{t^{s-1}}{e^{-t}-1}\, dt + 1 \right) = -\lim_{s \to 1} \left(\frac{\zeta(s)}{\Gamma(2-s)} - \frac{1}{s-1} \right) \tag{34}$$

Now suppose the limit in the question has value A, and expand $\zeta(s)/\Gamma(2-s)$ in a Laurent series about $s = 1$, to order 1:

$$\frac{\zeta(s)}{\Gamma(2-s)} = \left(\frac{1}{s-1} + A + o(1) \right) (1 + \gamma(s-1) + o(1))^{-1} = \frac{1}{s-1} + (A - \gamma) + o(1), \tag{35}$$

and so the only possibility that is consistent with the left-hand side of (34) is $A = \gamma$.

Finally, we wish to compute $\zeta'(0)$. The easiest way to do this is to use the functional equation:

$$\zeta(1-s) = 2(2\pi)^{-s} \cos(\pi s/2) \Gamma(s) \zeta(s) \tag{36}$$

If we let $s \to 1$, we find $\zeta(0) = -1/2$ using Laurent expansions. Having found this, it is much easier to work with the logarithmic derivative of (36), since sums are easier to compute than products. We have:

$$-\frac{\zeta'(1-s)}{\zeta(1-s)} = -\log 2\pi + \frac{\pi}{2} \tan(\pi s/2) + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}. \tag{37}$$

Taking $s \to 1$, the two terms that we have to worry about are the ζ and the tan. However, in fact we have

$$\frac{\pi}{2} \tan(\pi s/2) + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s} - \gamma - \frac{1}{s} + \gamma + o(s-1) \quad \text{as } s \to 1, \tag{38}$$

by a straightforward computation, so in fact $\zeta'(0) = -\frac{1}{2} \log 2\pi$ as required.