Further Complex Methods Sheet 3: Integral Transforms

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1 Definitions and Results

Everyone agrees that the definition of the Laplace transform is

$$(\mathcal{L}f)(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt. \tag{1}$$

The inverse is given by the *Bromwich integral*,

$$(\mathcal{L}^{-1}F)(t) = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts} F(s)\,ds, \tag{2}$$

where the contour is to the right of any singularities of F. In particular, we have

$$(\mathcal{L}f')(s) = s(\mathcal{L}f)(s) - f(0), \qquad (\mathcal{L}f'')(s) = s^2(\mathcal{L}f)(s) - sf'(0) - f(0) \tag{3}$$

and so on. Lastly, the convolution theorems are

$$(\mathcal{L}(f\cdot g))(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)G(s-z)\,dz \tag{4}$$

$$\left(\mathcal{L}\int_0^t f(\tau)g(t-\tau)\,d\tau\right)(s) = F(s)G(s). \tag{5}$$

On the other hand, there are an absurd number of different conventions for the Fourier transform. We take one that aligns with the Laplace transform:[1]

$$\tilde{f}(k) = (\mathcal{F}f)(k) = \int_{-\infty}^\infty e^{-ikx} f(x)\,dx \tag{6}$$

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx} g(k)\,dk. \tag{7}$$

The convolution theorems are then

$$\mathcal{F}(f * g) = \tilde{f}\cdot\tilde{g} \tag{8}$$

$$\mathcal{F}(f\cdot g) = \frac{1}{2\pi}\tilde{f} * \tilde{g} \tag{9}$$

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[1] This is not the author's preferred definition, due to the annoying factors of 2π that sneak up on you, especially in the convolution theorem.

2 Question 7

The system we are trying to solve is

$$y'' - k^2 y = f(t), \quad k > 0, \quad y(0) = y_0, \quad y'(0) = y_0'. \tag{10}$$

Taking the Laplace transform of the DE gives

$$(s^2 - k^2)Y(s) = F(s) + sy_0' + y_0, \tag{11}$$

or

$$Y(s) = \frac{F(s)}{s^2 - k^2} + y_0' \frac{s}{s^2 - k^2} + y_0 \frac{1}{s^2 - k^2} \tag{12}$$

The last two terms we recognise as the Laplace transforms of $\cosh kt$ and $k^{-1} \sinh kt$ respectively.[2] The first term is, of course, the Laplace transform of a convolution, as given above, and so we find that

$$y(t) = y_0 \cosh kt + \frac{y_0'}{k} \sinh kt + \int_0^t f(t') \frac{\sinh k(t' - t)}{k} \, dt' \tag{13}$$

as required.

If now we take $F(s) = (s + k_0)^{-1}$ in (12), we can calculate the inverse Laplace transform using partial fractions, since if $k \neq k_0$,

$$\frac{1}{(s + k_0)(s^2 - k^2)} = \frac{1}{k_0^2 - k^2} \left(\frac{1}{s + k_0} + \frac{-s + k_0}{s^2 - k^2} \right), \tag{14}$$

so we can for now concentrate on the bracket.

Now take the Bromwich integral. Since when $t > 0$, e^{ts} decays for $\Re(s) \to -\infty$, we can complete the contour with a large semicircle to the left. Applying Jordan's Lemma shows that the integral over the semicircle tends to zero as the radius tends to ∞, so we find that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} F(s) \, ds = \sum \text{Res}. \tag{15}$$

There are at this point three singularities: $s = \pm k$ and $s = -k_0$. The residue at $-k_0$ is simply e^{-sk_0}. At $\pm k$ the residue is

$$\frac{-(\mp k) + k_0}{\mp 2k} e^{\pm kt}, \tag{16}$$

and so adding everything up gives

$$y(t) = y_0 \cosh kt + \frac{y_0'}{k} \sinh kt + \frac{1}{k_0^2 - k^2} \left(e^{-k_0 t} - \cosh kt + \frac{k_0}{k} \sinh kt \right), \tag{17}$$

as expected. Checking that this agrees with the first part is a straightforward application of the hyperbolic addition formula and integration by parts in the usual way, so is omitted. There are various ways of doing the last part: one could evaluate the residue of the now double pole at $s = -k$, but it is perhaps simpler to series expand the solution about $k_0 = k$:

$$e^{-k_0 t} + \cosh\left((k - k_0)t + k_0 t\right) + \frac{k_0}{k} \sinh\left((k - k_0)t + k_0 t\right) \sim (k_0 - k) \left(-te^{-kt} + \frac{1}{k} \sinh kt \right), \tag{18}$$

and then dividing gives

$$y(t) = y_0 \cosh kt + \frac{y_0'}{k} \sinh kt + \frac{1}{2k} \left(-te^{-kt} + \frac{1}{k} \sinh kt \right). \tag{19}$$

[2] Or we could sweep the Bromwich contour to the left, picking up the residues at $\pm k$, and put the functions back together, but this is much less algebra!

3 Question 8

We want to solve

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad u(x,0) = f(x). \tag{20}$$

First, we take the Fourier transform, which naturally suggests the x variable, being over the whole line. We find

$$i\tilde{u}_t - k^2\tilde{u} = 0, \tag{21}$$

where of course $\tilde{u} = \tilde{u}(k,t)$. This is a first-order differential equation, which we know how to solve:

$$\tilde{u} = Ae^{-itk^2}, \tag{22}$$

and we find immediately upon setting $t = 0$ that $A = \tilde{f}$. Now we can apply the convolution theorem for the product of the two Fourier transforms:

$$u(x,t) = (\mathcal{F}^{-1}\tilde{u})(x,t) = (\mathcal{F}^{-1}e^{-itk^2}) * f; \tag{23}$$

we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-itk^2}\,dk = \frac{e^{-x^2/4it}}{2\pi}\int_{-\infty}^{\infty}\exp\left(-it\left(k-\frac{x}{2t}\right)^2\right)dk. \tag{24}$$

A shift and rescale of the integration variable gives

$$\frac{e^{-x^2/4it}}{2\pi\sqrt{t}}\int_{-\infty}^{\infty} e^{-iy^2}\,dy = \frac{1}{\sqrt{4\pi it}}e^{-x^2/4it}, \tag{25}$$

and hence the solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi it}}\int_{-\infty}^{\infty} e^{-(x-x')^2/4it} f(x')\,dx', \tag{26}$$

which looks rather like the heat equation solution.

Now, we consider the Laplace transform of this equation. Since the Laplace transform applies to initial value problems, we should transform with respect to t; doing this, we find

$$isU + U_{xx} = if, \tag{27}$$

which still looks pretty grisly, especially since f depends on x. We now have to find the Green's function for $\partial_x^2 + is$, i.e. solve

$$G_{xx} + isG = \delta(x-x'). \tag{28}$$

Write $\alpha = \sqrt{-is}$ for now. We need to choose the right branch of the square root, so that $\Re(\alpha)$ is always nonnegative; we can make such a choice by taking the principal branch of the square root, which causes α's branch cut to be along the negative imaginary axis. Then we can take a solution that decays as $x \to -\infty$:

$$G_-(x) = A\exp\left(\alpha(x-x')\right), \tag{29}$$

and the one that decays as $x \to +\infty$ which matches this is

$$G_+(x) = A\exp\left(-\alpha(x-x')\right), \tag{30}$$

and we then find A from the jump condition

$$G_+'(x') - G_-'(x') = 1. \tag{31}$$

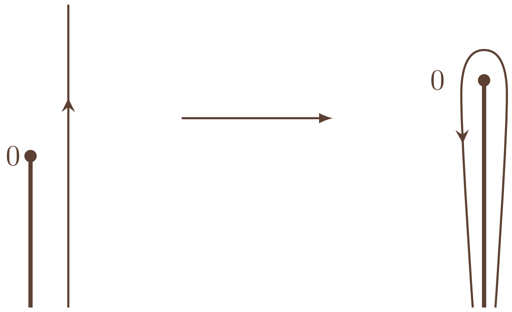

Figure 1: The deformation of the contour for the Bromwich integral

Then we find that

$$G(x;x') = -\frac{1}{2\alpha}\exp(-\alpha|x-x'|), \tag{32}$$

and so

$$U(x,s) = \int_{-\infty}^{\infty} \frac{-i}{2\alpha}\exp(-\alpha|x-x'|)f(x')\,dx' \tag{33}$$

Of course, now we have to get back to u. To do this, assume we can interchange the order of integration, and now we need to calculate the inverse Laplace transform of $\frac{-i}{\sqrt{-4is}}\exp\left(-\sqrt{-is}\,|x-x'|\right)$. For brevity write $X = |x-x'|$ The singularities are confined to the branch cut on the negative imaginary axis, and so we need to find

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{-i}{\sqrt{-4is}}\exp\left(ts-\sqrt{-is}X\right)ds \tag{34}$$

A Jordan's lemma-type argument allows us to deform the contour so that it wraps around the branch cut. The integrand is $O(s^{-1/2})$ at 0, so the contour can be made into a straight line on either side of the branch cut:

$$-\frac{1}{4\pi}\int_{\varepsilon-i\infty}^{0}\frac{1}{\sqrt{-is}}\exp\left(ts-\sqrt{-is}X\right)ds - \frac{1}{4\pi}\int_{0}^{-\varepsilon-i\infty}\frac{1}{\sqrt{-is}}\exp\left(ts-\sqrt{-is}X\right)ds \tag{35}$$

Now set $s = iy^2$, so $ds = 2iy\,dy$, $\sqrt{-is} = -y$ in the first integral and $+y$ in the second. Then the whole expression becomes

$$\frac{-i}{2\pi}\int_0^{\infty} e^{ity^2}\left(e^{-yX}+e^{yX}\right)dy = \frac{-i}{2\pi}\int_{-\infty}^{\infty} e^{ity^2-yX}\,dy, \tag{36}$$

by symmetry. We've done this sort of integral loads of times before: completing the square gives

$$ity^2 - yX = it\left(y-\frac{X}{2it}\right)^2 - \frac{X^2}{4it}, \tag{37}$$

and hence the integral is

$$e^{-X^2/4it}\frac{-i}{\pi}\int_{-\infty}^{\infty} e^{it(y-X/2it)^2}\,dt = \frac{-i}{\sqrt{-4\pi it}}e^{-X^2/4it} \tag{38}$$

using the result in the question. It is easy to check that $-i/\sqrt{-i} = 1/\sqrt{i}$, and so we finally once again obtain

$$u(x,t) = \frac{1}{\sqrt{4\pi it}}\int_{-\infty}^{\infty} e^{-(x-x')^2/4it}f(x')\,dx'. \tag{39}$$

Hooray!