Bessel Functions and Some of Their Many Identities[*][†]

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v.1 5 December 2014

1 Definition

Bessel's equation of order ν is, in Sturm–Liouville form,

$$-(zy')' + (z^{-1}\nu^2 - z)y = 0. \tag{1.1}$$

Using Frobenius gives a solution (the *Bessel function of the first kind*)

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(1+\nu+k)} \left(\frac{z}{2}\right)^{2k}, \tag{1.2}$$

where as far as we are concerned, $\Gamma(1+z) = z!$ for integer z, and everywhere else it retains the property that $z\Gamma(z) = \Gamma(1+z)$. When ν is not an integer, $J_{-\nu}$ is a linearly independent solution of Bessel's equation with the same ν. However, when $\nu = -n$ is a nonpositive integer, this is no longer true:

$$J_{-n}(z) = \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(1+k-n)} \left(\frac{z}{2}\right)^{2k} = \left(\frac{z}{2}\right)^{n} \sum_{k=n}^{\infty} \frac{(-1)^k}{k!\Gamma(1+k-n)} \left(\frac{z}{2}\right)^{2(k-n)} \tag{1.3}$$

because the fundamental relation for the Γ-function tells us that it diverges at negative integers, so the terms with $1+k-n<0$ all vanish. Reindexing by sending $k \mapsto k+n$ gives the relationship

$$J_{-n}(z) = (-1)^n J_n(z), \quad n \in \mathbb{Z}. \tag{1.4}$$

Therefore we need a linearly independent solution for $n \in \mathbb{Z}$. The following linear combination is the canonical choice:

$$Y_\nu(z) = J_\nu(z)\cot\nu\pi - J_{-\nu}(z)\operatorname{cosec}\nu\pi; \tag{1.5}$$

this is also known as the *Bessel function of the second kind* for obvious reasons. The interested reader will find that at nonnegative integers n,

$$Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\nu(z)}{\partial\nu} - (-1)^n \frac{\partial J_\nu(z)}{\partial\nu}\right]_{\nu=n}, \tag{1.6}$$

and the interested (and bored) reader can go on to find the series expansion,

$$Y_\nu(z) = \frac{2}{\pi}J_n(z)\log z - \frac{1}{\pi}\left(\frac{z}{2}\right)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k} - \frac{1}{\pi}\left(\frac{z}{2}\right)^{n}\sum_{k=0}^{\infty}\frac{\digamma(k+1)+\digamma(n+k+1)}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k}, \tag{1.7}$$

where $\digamma$, the *Digamma-function*, is the derivative of $\log\Gamma$. You can see why the series for Y_n's not waved around much.

(Note that most of the formulae given below apply equally to $Y_\nu(z)$, since it is a linear combination of $J_{\pm\nu}$ and also satisfies Bessel's equation.[1])

2 Derivative Identities

We shall show:

$$[z^{-\nu}J_\nu(z)]' = -z^{-\nu}J_{\nu+1}(z), \qquad [z^{\nu+1}J_{\nu+1}(z)]' = z^{\nu+1}J_\nu(z) \tag{2.1}$$

(well, we shall prove the first one, and leave the second as an easy exercise in the same vein).

$$\begin{aligned} z^\nu[z^{-\nu}J_\nu(z)]' &= z^\nu\left(\frac{1}{2}\right)^\nu \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(1+\nu+k)}\left(\frac{1}{2}\right)^{2k}\frac{d}{dz}(z^{2k}) = \left(\frac{z}{2}\right)^\nu \sum_{k=1}^{\infty}\frac{(-1)^k(2k)}{k!\Gamma(1+\nu+k)}\left(\frac{1}{2}\right)^{2k} z^{2k-1} \\ &= \left(\frac{z}{2}\right)^\nu \sum_{k=1}^{\infty}\frac{(-1)^k}{(k-1)!\Gamma(1+(\nu+1)+(k-1))}\left(\frac{z}{2}\right)^{2k-1} = \left(\frac{z}{2}\right)^\nu \sum_{r=0}^{\infty}\frac{-(-1)^r}{r!\Gamma(1+(\nu+1)+r)}\left(\frac{z}{2}\right)^{2r+1} = -J_{\nu+1}(z), \end{aligned}$$

[*] The interested reader is recommended G.N. Watson's famous chunky tome *A Treatise on the Theory of Bessel Functions* available from all good college libraries (unless someone's already got it out) or from Cambridge University Press (if you'd like to make your book grant disappear). Essentially all of this material is discussed at greater length, along with much much more.

[†] And since you're wondering, the font is Linux Libertine.

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[1] For the concerned: at integers, we can swap d/dz and $\partial/\partial\nu$ because $J_{\pm\nu}$ are continuous functions of ν.

as we wanted.

Bessel's equation can be derived from (2.1). If we write $\vartheta = z\frac{d}{dz}$, (2.1) become the simpler

$$(\vartheta - \nu)J_\nu(z) = -zJ_{\nu+1}(z), \qquad (\vartheta + \nu + 1)J_{\nu+1}(z) = zJ_\nu(z). \tag{2.2}$$

Eliminating $J_{\nu+1}$ from these gives

$$-zJ_\nu(z) = (\vartheta + \nu + 1)z^{-1}(\vartheta - \nu)J_\nu(z) = z^{-1}(\vartheta^2 - \nu^2)J_\nu(z), \tag{2.3}$$

because we can show that $(\vartheta + 1)z^{-1} = z^{-1}\vartheta$; this is equivalent to Bessel's equation after resubstituting for ϑ.

3 Indefinite Integral

As far as we are concerned, the most useful indefinite integral of Bessel functions is of the form

$$I := \int J_\nu(\alpha z)J_\nu(\beta z)z\,dz = \frac{z(\alpha J_{\nu+1}(\alpha z)J_\nu(\beta z) - \beta J_\nu(\alpha z)J_{\nu+1}(\beta z))}{\alpha^2 - \beta^2}, \quad \alpha \neq \beta. \tag{3.1}$$

First, note that the differential relations with z replaced by αz become

$$[z^{-\nu}J_\nu(\alpha z)]' = -\alpha z^{-\nu}J_{\nu+1}(\alpha z), \qquad [z^{\nu+1}J_{\nu+1}(\alpha z)]' = \alpha z^{\nu+1}J_\nu(\alpha z).$$

Writing the integrand as $J_\nu(\alpha z)J_\nu(\beta z)z = [z^{\nu+1}J_\nu(\alpha z)][z^{-\nu}J_\nu(\beta z)]$ and using integration by parts:

$$\int [z^{\nu+1}J_\nu(\alpha z)][z^{-\nu}J_\nu(\beta z)]\,dz = \left[\frac{z^{\nu+1}}{\alpha}J_{\nu+1}(\alpha z)\right][z^{-\nu}J_\nu(\beta z)] + \frac{\beta}{\alpha}\int [z^{\nu+1}J_{\nu+1}(\alpha z)][z^{-\nu}J_{\nu+1}(\beta z)]\,dz.$$

Refactoring the new integrand as $[z^{-\nu}J_{\nu+1}(\alpha z)][z^{\nu+1}J_{\nu+1}(\beta z)]$, we integrate by parts again:

$$\begin{aligned} I &= \frac{1}{\alpha}zJ_{\nu+1}(\alpha z)J_\nu(\beta z) + \frac{\beta}{\alpha}\left(-\frac{1}{\alpha}[z^{-\nu}J_\nu(\alpha z)][z^{\nu+1}J_{\nu+1}(\beta z)] + \frac{\beta}{\alpha}\int [z^{-\nu}J_\nu(\alpha z)][z^{\nu+1}J_\nu(\beta z)]\,dz\right) \\ &= \frac{\alpha zJ_{\nu+1}(\alpha z)J_\nu(\beta z) - \beta zJ_\nu(\alpha z)J_{\nu+1}(\beta z)}{\alpha^2} + \frac{\beta^2}{\alpha^2}I. \end{aligned}$$

Rearranging gives the result.

4 Orthogonality

We can use the previous integral to obtain

$$\int_0^1 J_\nu(\alpha z)J_\nu(\beta z)z\,dz = \frac{\alpha J_{\nu+1}(\alpha)J_\nu(\beta) - \beta J_\nu(\alpha)J_{\nu+1}(\beta))}{\alpha^2 - \beta^2}, \quad \alpha \neq \beta. \tag{4.1}$$

Then if $j_{\nu,k}$ is the kth positive root of the νth Bessel function, we immediately have

$$\int_0^1 J_\nu(j_{\nu,k}z)J_\nu(j_{n,l}z)z\,dz = 0, \quad k \neq l. \tag{4.2}$$

Obviously we run into problems with the $k = l$ case. We can mitigate this by setting $\alpha = j_{\nu,k} + \varepsilon$ and $\beta = j_{\nu,k}$ and taking the limit:

$$\int_0^1 J_\nu((j_{\nu,k}z)^2z\,dz = \lim_{\varepsilon\to 0}\frac{0 - j_{\nu,k}J_\nu(j_{\nu,k} + \varepsilon)J_{\nu+1}(j_{\nu,k}))}{(j_{\nu,k} + \varepsilon)^2 - j_{\nu,k}^2} = \lim_{\varepsilon\to 0} -\frac{j_{\nu,k}J'_\nu(j_{\nu,k} + \varepsilon)J_{\nu+1}(j_{\nu,k})}{2j_{\nu,k} + \varepsilon} = -\frac{J'_\nu(j_{\nu,k})J_{\nu+1}(j_{\nu,k})}{2}.$$

To put this in a simpler form, we would like to eliminate one of the functions on the right. Looking again at the differentiation formulae in the form (2.2) and setting $z = j_{\nu,k}$, we find

$$-j_{\nu,k}J_{\nu+1}(z) = j_{\nu,k}J'_\nu(j_{\nu,k}) + nJ_\nu(j_{\nu,k}) = 0, \tag{4.3}$$

so in particular, $J'_\nu(j_{\nu,k}) = -J_{\nu+1}(j_{\nu,k})$, and we have the orthogonality relation.

$$\int_0^1 J_\nu(j_{\nu,k}z)J_\nu(j_{n,l}z)z\,dz = \delta_{kl}\frac{J'_\nu(j_{\nu,k})^2}{2} = \delta_{kl}\frac{J_{\nu+1}(j_{\nu,k})^2}{2}. \tag{4.4}$$

5 And Finally...

The following formula holds (Watson, p. 411):

$$\int_0^\infty J_\nu(at)J_\nu(bt)J_\nu(ct)\frac{dt}{t^{\nu-1}} = \frac{2^{\nu-1}\Delta(a,b,c)^{2\nu-1}}{(abc)^\nu\Gamma(\nu + 1/2)\Gamma(1/2)}, \quad \Re(\nu) > -\tfrac{1}{2} \tag{5.1}$$

where $\Delta(a,b,c)$ is the area of the triangle with sides a, b and c. This is called *Sonine's Formula*, and is my candidate for the worst way to work out the area of a triangle.[2] It's even zero if no such triangle exists!

[2] I probably don't want to see alternative candidates.