Fourier Series
The Sine Product Formula
A Cotangent Series*

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1 A cosine Fourier series

Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. We start by doing something rather perverse: expanding $\cos\alpha x$ as a Fourier series on $[-\pi, \pi]$. This function will be continuous at π, since $\cos\alpha x$ is even, but not differentiable there, so we expect the coefficients to be $O(n^{-2})$ as $n \to \infty$. Obviously there will be no sine coefficients, so we need to know

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\alpha x \cos nx \, dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(\cos{(\alpha-n)x} + \cos{(\alpha+n)x}\right) \, dx \\ &= \frac{1}{\pi}\frac{1}{\alpha-n}\sin{(\alpha-n)\pi} + \frac{1}{\pi}\frac{1}{\alpha+n}\sin{(\alpha+n)\pi} \\ &= \frac{1}{\pi}(-1)^n\left(\frac{1}{\alpha-n}+\frac{1}{\alpha+n}\right)\sin\pi\alpha \\ &= \frac{(-1)^n}{\pi}\frac{2\alpha}{\alpha^2-n^2}\sin\pi\alpha. \end{aligned}$$

Therefore the Fourier series is

$$\cos\alpha x = \frac{2\alpha\sin\pi\alpha}{\pi}\left(\frac{1}{2\alpha^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2-n^2}\cos nx\right).$$

Setting $x = \pi$ and $\alpha = z$ and rearranging, we discover the extraordinary formula

$$\pi\cot\pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}.$$

Now, for $|z| \leqslant t$, the sum on the right satisfies

$$\left|\frac{1}{z^2-n^2}\right| \leqslant \frac{1}{n^2-t^2},$$

the latter of which converges using the integral test, so the Weierstrass M-test implies the sum converges uniformly to the function $\pi\cot\pi z - 1/z$ in $|z| \leqslant t < 1$. We can therefore integrate

$$\pi\cot\pi w - \frac{1}{w} = \sum_{n=1}^{\infty}\frac{2w}{w^2-n^2}$$

term-by-term from 0 to z to obtain

$$\log\left(\frac{\sin\pi z}{\pi z}\right) = \sum_{n=1}^{\infty}\log\left(1-\frac{z^2}{n^2}\right)$$

(notice that although this looks like something obscure has happened, it is a correct antiderivative of the cot expression with the correct value at 0, so must be correct). Exponentiating both sides, we have found using real analysis only that

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right),$$

an identity famously extrapolated from the polynomial case by Euler. (It is easy to derive the equivalence throughout $\mathbb{C}$ using periodicity, after checking it for the right-hand side.)

2 An integral and the cosecant formula

It is easy to prove analogously the formula for cosec: putting $x = 0$, we find

$$\begin{aligned} \pi\csc\pi z &= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z(-1)^n}{z^2-n^2} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{z+n}+\frac{1}{z-n}\right) \end{aligned}$$

We can use this to provide a more plausible proof of the value of the integral

$$I(s) = \int_0^{\infty}\frac{t^s}{1+t}\frac{dt}{t}, \quad 0 < s < 1$$

than is given by just doing a contour integral.[1]

Splitting the integral at 1 and using the substitution $u = 1/t$, we have

$$\int_1^{\infty}\frac{t^s}{1+t}\frac{dt}{t} = \int_0^1\frac{u^{-s}}{1+u^{-1}}\frac{dy}{u},$$

so

$$I(s) = \int_0^1\frac{t^s+t^{1-s}}{1+t}\frac{dt}{t}.$$

We can now use the binomial expansion:

$$\frac{1}{1+t} = \sum_{k=0}^{\infty}(-1)^k t^k,$$

* Please notice the title is a senryū …

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[1] This was, naturally, suggested by Hardy.

or

$$\frac{1}{1+t} = \sum_{k=0}^{n}(-1)^k t^k + (-1)^{n+1}\frac{t^{n+1}}{1+t}$$

Then if we write

$$I(s) = \int_0^1 \frac{t^{s-1}}{1+t}\,dt + \int_0^1 \frac{t^{-s}}{1+t}\,dt,$$

we find that the first integral becomes

$$\begin{aligned}\int_0^1 \frac{t^{s-1}}{1+t}\,dt &= \sum_{k=0}^{n}(-1)^k \int_0^1 t^{s+k-1}\,dt + (-1)^n \int_0^1 \frac{t^{s+n-1}}{1+t}\,dt \\ &= \sum_{k=0}^{n}(-1)^k \frac{1}{s+k} + (-1)^n \int_0^1 \frac{t^{s+n-1}}{1+t}\,dt \\ &= \frac{1}{s} + \sum_{k=1}^{n}(-1)^k \frac{1}{s+k} + (-1)^n \int_0^1 \frac{t^{s+n-1}}{1+t}\,dt,\end{aligned}$$

and using upper limit $n-1$ for the second,

$$\begin{aligned}\int_0^1 \frac{t^{-s}}{1+t}\,dt &= \sum_{k=0}^{n-1}(-1)^k \int_0^1 t^{-s+k}\,dt + (-1)^{n-1} \int_0^1 \frac{t^{-s+n-1}}{1+t}\,dt \\ &= \sum_{k=0}^{n-1}(-1)^k \frac{1}{-s+k+1} + (-1)^{n-1} \int_0^1 \frac{t^{-s+n-1}}{1+t}\,dt \\ &= \sum_{k=1}^{n}(-1)^k \frac{1}{s-k} - (-1)^n \int_0^1 \frac{t^{-s+n-1}}{1+t}\,dt,\end{aligned}$$

and so

$$I(s) = \frac{1}{s} + \sum_{k=1}^{n}(-1)^k\left(\frac{1}{s+k}+\frac{1}{s-k}\right) + (-1)^n \int_0^1 \frac{(t^s - t^{-s})t^{n-1}}{1+t}\,dt$$

Now, since $0 < t < 1$ and $s > 0$, $t^s < t^{-s}$, and the remaining integral is bounded by $2\int_0^1 t^{-s+n-1}\,dt = 1/(n-s) \to 0$ as $n \to \infty$, so we conclude

$$I(s) = \frac{1}{s} + \sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{s+k} + \frac{1}{s-k}\right).$$

But of course the right-hand side is $\pi \csc \pi s$, so so must be the integral.

I like to think of this as "the real reason"[2] that $I(s) = \pi \csc \pi s$. A similar computation can be done on any function that can be made symmetric under $x \mapsto 1/x$ (here, the function is in fact $1/(x^{1/2} + x^{-1/2})$, the extra $x^{1/2}$ being absorbed by the x^s).

Indeed, this result also generalises to complex values of s, but since this is a real argument in a real course, we shall leave the minor modifications necessary for this generalisation as an exercise to the reader.

[2]Pun not originally intended.