Fourier Series, the short guide

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Note

Practically everything on this sheet can be proved by swapping summation and integration (assuming it is okay to do so ...).

1 Trigonometrical Fourier Series

Assume throughout that $f(x)$ is real-valued.

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = L\delta_{mn} \tag{1.1}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = L\delta_{mn} \tag{1.2}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0 \tag{1.3}$$

(Proof: use the prosthaphaeresis formulae $\cos m\theta \cos n\theta = \frac{1}{2}[\cos(m-n)\theta + \cos(m+n)\theta]$ &c.)

1.1 Fourier Series

Defining the *Fourier coefficients* of a periodic function $f(x)$, with $f(x+2L) = f(x)$, as

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx \tag{1.4}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \tag{1.5}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \tag{1.6}$$

for sufficiently nice functions,[1]

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \tag{1.7}$$

Note that at a discontinuity, the Fourier series takes on the value

$$\lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) + f(x-\varepsilon)}{2}.$$

1.2 Parseval's Theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \tag{1.8}$$

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[1] $f(x)$ piecewise continuous with a finite number of finite discontinuities, maxima and minima is sufficient.

2 Fourier Cosine Series

A function on $[0, L]$ can be extended to an even function on $[-L, L]$ by $f(-x) = f(x)$ for $-L < x < 0$. Such a function has a Fourier cosine series,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{2.1}$$

with coefficients given by

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \tag{2.2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \tag{2.3}$$

3 Fourier Sine Series

A function on $[0, L]$ can be extended to an odd function on $[-L, L]$ by $f(-x) = -f(x)$ for $-L < x < 0$. Such a function has a Fourier sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{3.1}$$

with coefficients given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{3.2}$$

4 Complex Fourier Series

We now consider instead the functions $\exp(in\pi x/L)$. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ shows that what we are doing will end up equivalent to our original version.

4.1 Orthogonality relation

Only one (but notice the signs!):

$$\int_{-L}^{L} e^{im(\pi x/L)}e^{-in(\pi x/L)}\,dx = 2L\delta_{mn} \tag{4.1}$$

4.2 Fourier series

Therefore, with

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in(\pi x/L)}\,dx \tag{4.2}$$

we have the complex Fourier series expansion for $f(x)$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in(\pi x/L)} \tag{4.3}$$

Further, the relationship between a_n and b_n and the c_n is, using Euler's formula,

$$c_0 = \frac{1}{2}a_0, \qquad c_n = \frac{1}{2}(a_n - ib_n), \qquad c_{-n} = \frac{1}{2}(a_n + ib_n). \tag{4.4}$$

$(n > 0)$ or vice versa:

$$a_0 = 2c_0, \qquad a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n}). \tag{4.5}$$

4.3 Parseval's Theorem

$$\frac{1}{2L}\int_{-L}^{L}[f(x)]^2\,dx = \sum_{n=1}^{\infty}|c_n|^2 \tag{4.6}$$