The Fourier Transform

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1 Definition

There are about 8 possible conventions for the Fourier Transform.[1] We shall use the definition from lectures (or at least, the definition from my lectures ...). Given $f : \mathbb{R} \to \mathbb{C}$, the *Fourier transform* of f is a function $\tilde{f} : \mathbb{R} \to \mathbb{C}$ given by

$$\tilde{f}(k) := \mathcal{F}[f](k) := \mathcal{F}[f(x)](k) := \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx. \tag{1.1}$$

The *inverse Fourier transform* is defined as

$$\mathcal{F}^{-1}[g](x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} g(k)\, dk. \tag{1.2}$$

Then (key, cool theorem): these two transforms are the inverses of each other[2]

$$f(x) = \mathcal{F}^{-1}[\tilde{f}](x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k)\, dk. \tag{1.3}$$

2 The key properties

Let $a, b, k_0 \in \mathbb{C}$ and $c, x_0 \in \mathbb{R}$ be constants.

$$(f * g)(x) := \int_{-\infty}^{\infty} f(y) g(x-y)\, dy \tag{2.1}$$

is the *convolution of f and g* and $f \cdot g$ denotes pointwise multiplication.

A table of most of the most commonly used properties is found overleaf.

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[1] There are 4 places to put the 2π: one in the exponential, all of it in the inverse, all of it in the forward transform, or half in each, and two sign conventions in the exponential, $4 \times 2 = 8$. There are also a number of different conventions for the notation for $\mathcal{F}[f]$: $\tilde{f}(k)$, $\hat{f}(k)$, $f(k)$ (for the truly perverse), $\mathcal{F}f$ &c., and k can also be ξ, ω or ν &c. The author's preference is for the $\sim$ since it's wavey, like e^{ikx}. See also §5.

[2] *cough* On a certain space of integrable functions *cough* You get the best results when f is integrable and square-integrable, $\int_{\mathbb{R}} |f| < \infty$ and $\int_{\mathbb{R}} |f|^2 < \infty$. As is customary in Applied courses, I'm not going to mention convergence again: those of you are worried need to either read Tom Körner's book or go to Part II Linear Analysis.

Property name	Function, $f(x)$	Fourier transform, $\tilde{f}(k)$	
Linearity	$af(x)+bg(x)$	$a\tilde{f}(k)+b\tilde{g}(k)$	(2.2)
Translation	$f(x-x_0)$	$e^{-ikx_0}\tilde{f}(k)$	(2.3)
	$e^{iaxk_0}f(x)$	$\tilde{f}(k-k_0)$	(2.4)
Scaling	$f(cx)$	$\frac{1}{\lvert c\rvert}\tilde{f}\left(\frac{k}{c}\right)$	(2.5)
Conjugation	$\overline{f(x)}$	$\overline{\tilde{f}(-k)}$	(2.6)
Derivatives and Multiplication	$f^{(n)}(x)$	$(ik)^n\tilde{f}(k)$	(2.7)
	$x^n f(x)$	$i^n\frac{d^n}{dk^n}\tilde{f}(k)$	(2.8)
Convolution	$(f*g)(x)$	$\frac{1}{2\pi}\tilde{f}(k)\cdot\tilde{g}(k)$	(2.9)
	$f(x)\cdot g(x)$	$(\tilde{f}*\tilde{g})(k)$	(2.10)

2.1 Parseval/Plancherel

$$\int_{-\infty}^{\infty}\overline{f(x)}g(x)\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\tilde{f}(k)}g(k)\,dk \tag{2.11}$$

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\tilde{f}(k)|^2\,dk \tag{2.12}$$

3 Poisson Summation Formula[3]

Suppose f and $\tilde{f}$ decay sufficiently fast at infinity.[4] Then

$$\sum_{n=-\infty}^{\infty}f(x+2\pi n)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}e^{ikx}\tilde{f}(k) \tag{3.1}$$

and in particular, we have the *Poisson Summation Formula*,

$$\sum_{n=-\infty}^{\infty}f(2\pi n)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\tilde{f}(k) \tag{3.2}$$

The proof is done simply by looking at the Fourier series of the function $\sum_{n=-\infty}^{\infty}f(x+2\pi n)$ on the interval $(-\pi,\pi)$.

A special case is the following:

[3]Remark: I've seen a Tripos question on this....
[4]Faster than $t^{-1-\varepsilon}$ will do.

Theorem 1 (Jacobi's imaginary transformation for the ϑ-function). *Let $z, \tau \in \mathbb{C}$, with $\Im(\tau) > 0$. Then*

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z} \tag{3.3}$$

converges to a periodic function with period 1, *and*

$$\vartheta\left(\frac{z}{\tau}; \frac{-1}{\tau}\right) = (-i\tau)^{1/2} e^{\pi i z^2/\tau} \vartheta(z;\tau). \tag{3.4}$$

One proof uses Fourier series and the general form of the Poisson Summation Formula. (Exercise: do it!) This is used in the theory of the Riemann zeta-function, as well as that of the *Jacobi Elliptic Functions*, which are defined as quotients of this function and its translates (see Part II Further Complex Methods, depending on the lecturer...).

4 A Very Short Table of Common Transform Pairs

Note With our convention for the Fourier Transform,

$$\mathcal{F}[\mathcal{F}[f]](x) = 2\pi f(-x), \tag{4.1}$$

so in a sense the table can be used "in both directions".

δ is the Dirac delta function, H the Heaviside step function, J_0 the zeroth order Bessel function of the first kind. Also, let

$$\operatorname{rect}(x) = H(1/2 + x) - H(1/2 - x) = \begin{cases} 1 & |x| \leqslant 1/2 \\ 0 & \text{else} \end{cases} \tag{4.2}$$

$f(x)$	$\tilde{f}(k)$	
1	$\delta(k)$	(4.3)
e^{iax}	$\delta(k-a)$	(4.4)
$e^{-a\|x\|}$	$\dfrac{2a}{k^2+a^2}$	(4.5)
e^{-ax^2}	$\sqrt{\dfrac{\pi}{a}} e^{-k^2/(4a)}$	(4.6)
$\cos ax$	$\frac{1}{2}(\delta(k-a)+\delta(k+a))$	(4.7)
$\sin ax$	$\frac{1}{2i}(\delta(k-a)-\delta(k+a))$	(4.8)
$\operatorname{sech} ax$	$\dfrac{\pi}{a}\operatorname{sech}\left(\dfrac{\pi k}{2a}\right)$	(4.9)
$H(x)$	$\dfrac{1}{ik} + \pi\delta(k)$	(4.10)
$\operatorname{rect}(ax)$	$\dfrac{2}{k}\sin\left(\dfrac{k}{2a}\right)$	(4.11)
$J_0(ax)$	$\dfrac{2\operatorname{rect}(k/2)}{\sqrt{1-k^2}}$	(4.12)

5 Other conventions

The following is a table of Fourier transform pairs. All Fourier transforms and their inverses are of type

$$\mathcal{F}_K[f](k) = \int_{-\infty}^{\infty} K_f(x) f(x)\, dx \qquad \mathcal{F}_K^{-1}[f](k) = \int_{-\infty}^{\infty} K_b(x) f(x)\, dx, \tag{5.1}$$

and the following table lists standard conventions for K_f and K_b. For a boring exercise, go and derive the basic properties for these conventions (or look on Wikipedia).[5]

$K_f(x)$	$K_b(x)$	Comments
e^{-ikx}	$\frac{1}{2\pi} e^{ikx}$	Convention in this document. Good for derivatives.
e^{ikx}	$\frac{1}{2\pi} e^{-ikx}$	Swapping the minus sign over. No real advantage.[6]
$\frac{1}{2\pi} e^{-ikx}$	e^{ikx}	2π in the forward transform.
$\frac{1}{\sqrt{2\pi}} e^{-ikx}$	$\frac{1}{\sqrt{2\pi}} e^{ikx}$	$\sqrt{2\pi}$ in both
$e^{-2\pi ikx}$	$e^{2\pi ikx}$	2π in the exponential. Used in signal processing. Poor for derivatives. Has the property that $\mathcal{F}^4[f] = f$.[8]
$\sqrt{\frac{\lvert b\rvert}{(2\pi)^{1-a}}} e^{ibkx}$	$\sqrt{\frac{\lvert b\rvert}{(2\pi)^{1+a}}} e^{-ibkx}$	Most general form

6 And Finally: Fourier Transform of a Spherically Symmetric Function

Those pesky Bessel functions come back to haunt us: in n dimensions, if the function f is just $f(|x|)$ and therefore spherically symmetric, we rewrite it using θ as the angle between x and k:

$$\tilde{f}(k) = \int_{\mathbb{R}^n} e^{-ik\cdot x} f(|x|)\, d^n x = \int_0^{\infty} f(r) r^{n-1} \left(\int_{S^{n-1}} e^{-i|k|r\cos\theta}\, d\Omega \right) dr = \int_0^{\infty} f(r) r^{n-1} K(|k|\, r)\, dr, \tag{6.1}$$

where $K(x)$ is the the integral of $e^{-ix\cos\theta}$ over the $(n-1)$-sphere. It so happens that this is expressible as

$$K(kr) := \int_{S^{n-1}} e^{-ikr\cos\theta} \sin^n\theta\, d\Omega = (2\pi)^{n/2} (kr)^{1-n/2} J_{n/2-1}(kr), \tag{6.2}$$

This is close to the *Hankel transform*, which is like the Fourier transform, but uses Bessel functions. Further, if we had used the correct convention (putting the 2π in the exponential), it would be its own inverse.

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[5] The ones that require paying most attention are the Convolution Theorem and Parseval/Plancherel.

[6] Or complex advantage.[7]

[6] "[...]Ha ha! Amusing in a quiet way," said Eeyore, "but not really helpful."

[8] My current favourite book on the Fourier transform (apart from Tom Körner's, of course) is Dym and McKean's *Fourier Series and Integrals* (Academic Press, 1985), which uses this convention (for good reason).