Separation of Variables

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We shall work with the usual co-ordinate systems and their labels:

- Cartesians: (x, y, z)
- Polar: (r, θ), converted to Cartesians as $(x, y) = (r \cos\theta, r \sin\theta)$
- Cylindricals: (r, θ, z), Cartesians being $(x, y, z) = (r \cos\theta, r \sin\theta, z)$
- Sphericals: (r, θ, ϕ), Cartesians being $(x, y, z) = (r \cos\phi \sin\theta, r \sin\phi \sin\theta, r \cos\theta)$.

Two-dimensional polars are essentially a special case of cylindricals with no z-dependence.

1 The Laplacian in Various Co-ordinate Systems

You showed in Vector Calculus the following:

- In Cartesians (x, y, z),
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{1.1}$$
- In Cylindricals (r, θ, z),
$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{1.2}$$
(2D Polars are just cylindricals with the z term missing.)
- In Sphericals (r, θ, ϕ),
$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{1.3}$$

2 General idea of SepVar

The main idea is that no matter what dirty trick we use to find the solution to a linear partial differential equation, as long as we can make it satisfy the boundary conditions it's a perfectly good solution. We therefore use the following algorithm:

1. We guess that we can write the solution function in the form
$$f(x, y, z) = X(x)Y(y)Z(z) \tag{2.1}$$
(it may be necessary to sum up several of these solutions, but this is fine since the equation is linear).[1]
2. We then rearrange the partial differential equation into a sum of terms, each of which is a function of only one variable. This says each term must be equal to constant, which we call the separation constants, λ, μ, &c.
3. We solve the ordinary differential equation for each term in general
4. We apply some of the boundary conditions (or continuity or similar) to determine what values the separation constants λ, μ, &c. can take.
5. We sum over all possible values of the separation constants λ, μ, &c.
6. We use eigenfunction expansions to make everything match up at the boundaries we haven't dealt with.

In the next section we shall deal mostly with Laplace's equation, since it's a fairly small step from there to the heat and wave equations.

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[1] Sometimes it is also beneficial to consider solutions of the form $f(x, y, z) = X(x) + Y(y) + Z(z)$, an additive separation rather than a multiplicative one.

3 Cartesian Co-ordinates

This is fairly straightforward. Setting $f(x, y, z) = X(x)Y(y)Z(z)$, Laplace's equation becomes

$$0 = X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z),$$

and dividing through, we get

$$0 = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{3.1}$$

Since each term on the right is a function of a different variable, this can only occur when

$$\frac{X''}{X} = -\mu, \qquad \frac{Y''}{Y} = -\nu, \qquad \frac{Z''}{Z} = \mu + \nu.$$

Two of these equations have negative numbers on the right, and one positive. Suppose $\mu, \nu > 0$, then the equations are

$$X'' + \mu X = 0, \qquad Y'' + \nu Y = 0, \qquad Z'' - (\mu + \nu)Z = 0, \tag{3.2}$$

which have solutions

$$X = A \cos \sqrt{\mu}x + B \sin \sqrt{\mu}x, \qquad Y = C \cos \sqrt{\nu}y + D \sin \sqrt{\nu}y \qquad Z = Ee^{\sqrt{\mu+\nu}z} + Fe^{-\sqrt{\mu+\nu}z}; \tag{3.3}$$

the possible values of all of the constants must be determined by the boundary conditions (but notice that x, y and z were to an extent equivalent, so it may be necessary to choose a different one of X, Y and Z to have exponentials to satisfy the boundary conditions).

4 Cylindrical Co-ordinates

Separate variables in (1.2):

$$f(r, \theta, z) = R(r)\Theta(\theta)Z(z); \tag{4.1}$$

the same division procedure as before gives us

$$\frac{1}{r^2}\left(\frac{r}{R}(rR')' + \frac{\Theta''}{\Theta}\right) + \frac{Z''}{Z} = 0. \tag{4.2}$$

This time we are not so fortunate: the second term is a function of z only, but the first bracket is a function of both r and θ. Supposing that

$$Z'' = k^2 Z \tag{4.3}$$

(this is conventional, since the Z equation is second-order), we can then rewrite (4.2) as

$$\left(\frac{r}{R}(rR')' - r^2k^2\right) + \frac{\Theta''}{\Theta} = 0. \tag{4.4}$$

Now the first term is a function of r only, the second a function of θ only, and so we can suppose they are constant, which gives us

$$\frac{r}{R}(rR')' - r^2k^2 + \mu = 0, \qquad \frac{\Theta''}{\Theta} = -\mu, \qquad \frac{Z''}{Z} = k^2, \tag{4.5}$$

The Θ equation is solved by

$$\Theta = Ae^{i\sqrt{\mu}\theta} + Be^{-i\sqrt{\mu}\theta}.$$

Now, $\theta = 0$ and $\theta = 2\pi$ describe the same point, so for the sake of continuity of Θ and its derivatives we should have $\Theta(\theta + 2\pi) = \Theta(\theta)$. This imposes the condition $\mu = m^2, m \in \mathbb{Z}$. Putting this into the equations and rearranging,

$$r(rR')' + (m^2 - r^2\lambda)R = 0, \qquad \Theta'' + m^2\Theta = 0, \qquad Z'' = k^2 Z. \tag{4.6}$$

The Z equation is then solved by

$$Z = Ae^{kz} + Be^{-kz}. \tag{4.7}$$

Now, depending on the nature of k, this may be exponential ($k^2 > 0$), or oscillatory ($k^2 < 0$). We normally have k real (but see later note).

The R equation becomes

$$r(rR')' + (m^2 - k^2 r^2)R = 0.$$

Substituting $z = kr$ gives

$$z(zR')' + (m^2 - z^2)R = 0; \tag{4.8}$$

this is *Bessel's equation of order* m. The solutions are called *Bessel functions*. There are two linearly independent solutions, $J_m(z)$, which is regular at $z = 0$, and $Y_m(z)$, which has at least a logarithmic singularity at $z = 0$. Both are oscillatory and have infinitely many simple zeros for positive z. See also separate handout. On a finite domain, we normally therefore want $k^2 > 0$.

Therefore the solution is some sort of sum of solutions of the form

$$(AJ_m(kr) + BY_m(kr))(Ce^{kz} + De^{-kz})e^{im\theta}, \tag{4.9}$$

over $m \in \mathbb{Z}$ and k in some set that is determined by the boundary conditions.

4.1 Note on possible types of solution

If $k^2 < 0$, the solution can be written in terms of the *modified Bessel functions*, which are the same as the ordinary sort with imaginary arguments. These do not have zeros on the positive real axis, however, and both types are not regular at the origin. This also forces Z to oscillate rather than have exponential behaviour,[2] so this solution also only tend to be relevant with periodic boundary conditions in the z-direction, in infinite domains excluding the origin. So in summary we normally have:

- Finite domain including the origin (i.e. a disk): $e^{\pm kz} J_m(kr)$.
- Finite domain not including the origin (i.e. an annulus): both $e^{\pm kz} J_m(kr)$ and $e^{\pm kz} Y_m(kr)$.
- Periodic boundary conditions in z, not including the origin: $e^{\pm ikz} K_m(kr)$.

There is another possibility, that the boundary conditions on Z permit $k = 0$. These terms are given by the solutions to Laplace's equation in polar co-ordinates, as discussed in the next section.

5 Other equations from Cylindrical Co-ordinates

5.1 Laplace's Equation in Polar Co-ordinates

We can think of this as there being no Z dependence, i.e. $Z \equiv 1$. Therefore k is simply not present, and the equations (4.6) become

$$r(rR')' + m^2 R = 0, \qquad \Theta'' + m^2 \Theta = 0. \tag{5.1}$$

We have already solved the equation for Θ. Notice that the R equation only contains the operator $r\frac{d}{dr}$. The eigenfunctions of this operator are r^k, and substituting this in gives the "indicial" equation

$$k^2 = m^2,$$

and so for $m \neq 0$ the R corresponding to Θ is $Ar^k + Br^{-k}$. For $m = 0$, we get a solution of the form

$$R = A + B\log r. \tag{5.2}$$

Therefore the general solution to Laplace's equation in polar co-ordinates, taking into account all of the possibilities for m, is of the form

$$f(r,\theta) = A_0 + B_0 \log r + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (A_m r^m + B_m r^{-m})e^{im\theta}, \tag{5.3}$$

or perhaps a better form is

$$f(r,\theta) = A_0 + B_0 \log r + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} r^m (c_m e^{im\theta} + d_m e^{-im\theta}) = A_0 + B_0 \log r + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} r^m (a_m \cos m\theta + b_m \sin m\theta). \tag{5.4}$$

[2] As you have probably noticed by now, oscillatory behaviour in some of the co-ordinates, and exponential decay in the others, is a feature of Laplace's equation.

5.2 The Diffusion Equation in Polar Co-ordinates

The Diffusion equation in polar co-ordinates is

$$0 = -\frac{1}{D}\frac{\partial f}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \tag{5.5}$$

This is like (1.2), but with the double z derivative replaced by the single t derivative$-D^{-1}\partial/\partial t$. Separating variables via $f(r,\theta,t) = R(r)\Theta(\theta)T(t)$, (4.6) becomes

$$r(rR')' + (m^2 - k^2r^2)R = 0, \qquad \Theta'' + m^2\Theta = 0, \qquad T' + k^2DT = 0. \tag{5.6}$$

We notice that the R and Θ equations are the same as before. Again, on any finite domain we need $k^2 > 0$ so that the solutions to Bessel's equation have to have zeros on the positive real axis. Likewise if we are including the origin we cannot have the Y_m hanging around making our solution singular at $r = 0$. Therefore the T equation has the solution

$$T = Ae^{-k^2Dt},$$

and the whole solution contains a sum over k and m of terms of the form

$$(AJ_m(kr) + BY_m(kr))(Ce^{kz} + De^{-kz})e^{im\theta - k^2t}, \tag{5.7}$$

with $m \in \mathbb{Z}$ and k determined by the boundary conditions. (With possible polar-like additional terms due to the possibility of $k = 0$, as discussed in the previous section. We should check in individual problems that such terms do not appear.)

6 Laplace's Equation in Sphericals

Setting $f(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$, we can divide through in (1.3) in the same way as before to obtain

$$\sin^2\theta\left(\frac{(r^2R')'}{R} + \frac{(\Theta'\sin\theta)'}{\Theta\sin\theta}\right) + \frac{\Phi''}{\Phi} = 0 \tag{6.1}$$

As usual, we set

$$-\frac{\Phi''}{\Phi} = m^2 = \sin^2\theta\left(\frac{(r^2R')'}{R} + \frac{(\Theta'\sin\theta)'}{\Theta\sin\theta}\right), \tag{6.2}$$

since we require periodicity in ϕ, and get solutions

$$\Phi = Ae^{im\phi} + Be^{-im\phi}. \tag{6.3}$$

Then we can rearrange (6.2) to obtain

$$-\frac{(r^2R')'}{R} = \frac{(\Theta'\sin\theta)'}{\Theta\sin\theta} - \frac{m^2}{\sin^2\theta} = \lambda, \tag{6.4}$$

say. Then the equations become

$$\sin\theta(\Theta'\sin\theta)' - (m^2 - \lambda\sin^2\theta)\Theta = 0, \qquad (r^2R')' + \lambda R = 0. \tag{6.5}$$

For the Θ equation, substituting $z = \cos\theta$ and $P = \Theta(\arccos z)$ gives $d/d\theta = -\sin\theta d/dz$, and the equation becomes

$$\left((1-z^2)P'\right)' + \left(\lambda - \frac{m^2}{1-x^2}\right)P = 0. \tag{6.6}$$

This is called the *associated Legendre equation*. Tedious Frobenius computation shows that in order to get convergence at $z = 1$, we need the series to terminate, which requires $\lambda = \ell(\ell+1)$ where ℓ is a non-negative integer. The normalised polynomial solutions to this equation are the *associated Legendre polynomials*, $P_\ell^m(z)$.[3] We would also discover that we have to take $|m| \leqslant \ell$ to get a nonzero solution. As far as the R equation is concerned, staring at it reveals that we can write it as

$$(rR)'' = \ell(\ell+1)rR. \tag{6.7}$$

Substituting $R = r^k$ implies that $k(k+1) = \ell(\ell+1)$, which has solutions $k = \ell$ and $k = -\ell-1$,[4] so the full expansion is of the form

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(A_{\ell,m}r^\ell + B_{\ell,m}r^{-\ell-1})e^{im\theta}P_\ell^m(\cos\theta); \tag{6.8}$$

the axisymmetric restriction of this being

$$\sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-\ell-1})P_\ell(\cos\theta). \tag{6.9}$$

[3] Taking $m = 0$ gives the ordinary Legendre polynomials with which you are more familiar.
[4] And these are never equal for integer k, so no extra solutions appear with $k = 0$ for once!