Similarity Solution to the Diffusion Equation

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Let Φ be a function of x and t. The diffusion equation is

$$\partial_t \Phi - K \partial_x^2 \Phi = 0, \tag{1}$$

where K is some constant with dimensions of area/time.

A *similarity solution* to a differential equation is one in which in some way "the boundary conditions don't matter". What this means in practice is taking a function of (normally two) variables that satisfies a partial differential equation, and producing a function (of one variable) of some dimensionless combination of them.[1] There are in principle two ways to go about producing such a thing: use the differential equation to cook up a dimensionless combination, then impose scaling on Φ such that it still satisfies the differential equation, or a more careful technique using scaling invariance.

1 Bashing the Partial Differential Equation

Looking at the dimensions of things in the PDE, we have

$$[K][\Phi][x]^{-2} = [\Phi][t]^{-1}. \tag{2}$$

Therefore, the combination $\eta := x/\sqrt{Kt}$ is dimensionless:

$$[\eta^2] = [x]^2[D]^{-1}[t]^{-1} = 1. \tag{3}$$

Dimensionless quantities are normally important for telling us stuff about the system.[2] Therefore we try and find a differential equation in the variable η. Suppose

$$\Phi(x,t) = t^{-p} F\left(\frac{x}{\sqrt{Kt}}\right) = t^{-p} F(\eta). \tag{4}$$

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[1] This is based on the idea that dimensionless quantities are the fundamentals of the system. This reappears, particularly in Fluids, with things like the Reynolds number (which describes the extent to which a system is influenced by inertia compared to forces). A similar idea occurs in Quantum Field Theory, in things like the fine structure constant, α, which is a dimensionless parameter well-known to be close to, but not equal to, $1/137$: it is this that tells us how complicated our Feynman diagrams can become before we should stop adding them up to get a good approximation to the probability amplitude. (Answer: very. Feynman diagrams with 100-odd internal vertices are impossibly complicated to deal with using the current theory.)

[2] Yes, this is vague.

Then this Φ satisfies the diffusion equation (1) if and only if

$$\begin{aligned} 0 = \partial_t \Phi - K\partial_x^2 \Phi &= -\frac{p}{t} t^{-p} F + (\partial_t \eta) t^{-p} F' - K\left((\partial_x^2 \eta) t^{-p} F' + (\partial_x \eta)^2 t^{-p} F''\right) \\ &= -\frac{p}{t^{p+1}} F - \frac{\eta}{2t^{p+1}} F' - \frac{1}{t^{p+1}} F''. \end{aligned}$$

In other words, we want to solve the equation

$$2F'' + \eta F' + 2pF = 0, \tag{5}$$

choosing p sensibly.[3] Choosing $p = 1/2$ in (5) gives us

$$(2F' + \eta F)' = 0, \tag{6}$$

and the general solution to this equation is easily seen to be

$$F(\eta) = Ae^{-\eta^2/4} + Be^{-\eta^2/4} \int_0^\eta e^{t^2/4}\, dt. \tag{7}$$

Choosing $F'(0) = 0$ gives the solution from lectures.

2 Scaling

Alternatively, we consider looking for scalings that leave the form of the diffusion equation fixed.

Suppose Φ satisfies (1). Let $X = \lambda^a x$ and $T = \lambda^b t$, where λ is a function of t. Consider

$$\Phi_\lambda(X, T) := \lambda^c \Phi(X, T) = \lambda^c \Phi(\lambda^a x, \lambda^b t). \tag{8}$$

It turns out that what we are really interested in is what happens to the operator

$$\partial_t - K\partial_x^2. \tag{9}$$

To elaborate, first suppose λ is independent of t. Then:

$$\begin{aligned} (\partial_t - K\partial_x^2)\Phi_\lambda(X, T) &= (\partial_t - K\partial_x^2)\lambda^c \Phi(\lambda^a x, \lambda^b t) \\ &= \lambda^c(\lambda^b \partial_1 - \lambda^{2a} K \partial_2^2)\Phi(\lambda^a x, \lambda^b t) \\ &= \lambda^{c+b}(\partial_T - \lambda^{2a-b}\partial_X^2)\Phi(X, T), \end{aligned}$$

where the notation $\partial_1 \Phi(y_1, y_2)$ means $\frac{d}{d\epsilon}\big|_{\epsilon=0} \Phi(y_1 + \epsilon, y_2)$ and so on. Therefore, if and only if $2a = b$, the operators

$$(\partial_t - K\partial_x^2), \qquad\qquad (\partial_T - K\partial_X^2)$$

[3] This is very close to Hermite's differential equation, which you encounter in the quantum mechanics of the harmonic oscillator as

$$H'' - 2xH' + 2nH = 0;$$

in fact, (5) has one solution

$$F = e^{-\eta^2/4} H_{2p-1}(\eta/2),$$

H_n being the Hermite function with parameter n.

are proportional when acting on Φ, and in particular, relabelling $X \mapsto x$, $Y \mapsto y$, we have

$$(\partial_t - K\partial_x^2)\lambda^c \Phi(\lambda^a x, \lambda^{2a} t) = 0 \iff (\partial_t - K\partial_x^2)\Phi(x,t) = 0. \tag{10}$$

Now, suppose λ is instead a function of t. Now the diffusion equation becomes

$$\begin{aligned}
0 &= (\partial_t - K\partial_x^2)\Phi_\lambda(X,T) \\
&= (\partial_t - K\partial_x^2)\lambda^c \Phi(\lambda^a x, \lambda^b t) \\
&= \lambda^c \left(\frac{c\dot{\lambda}}{\lambda}\Phi + \partial_t \Phi \right) - K\lambda^{c+2a}\partial_X^2 \Phi(X,T) \\
&= \lambda^c \left(\frac{c\dot{\lambda}}{\lambda}\Phi + xa\dot{\lambda}\lambda^{a-1}\partial_1\Phi + \lambda^b \left(1 + \frac{\dot{\lambda}}{\lambda} bt \right) \partial_2 \Phi \right) - K\lambda^{c+2a}\partial_X^2 \Phi(X,T) \\
0 &= \lambda^c \left(\frac{c\dot{\lambda}}{\lambda}\Phi + \frac{a\dot{\lambda}}{\lambda} X\partial_X \Phi + \lambda^b \left(1 + \frac{\dot{\lambda}}{\lambda} bt \right) \partial_T \Phi \right) - K\lambda^{c+2a}\partial_X^2 \Phi(X,T),
\end{aligned} \tag{11}$$

the second term working in exactly the same way as for λ constant.

The point of all this is that if we make the correct choice of λ, we can make the coefficient of ∂_T go away and be left with an ordinary differential equation in the variable X. In other words, we have to make λ solve

$$1 + \frac{\dot{\lambda}}{\lambda} bt = 0. \tag{12}$$

Dividing through by t and integrating, we find

$$\begin{aligned}
0 &= \log A + \log t + b \log \lambda, \\
\lambda^b &= \frac{1}{At},
\end{aligned}$$

where A is a constant. Unsurprisingly, this is exactly the choice that makes T a constant, so we can write $\Phi(X,T) = F(X)$. Substituting this and $\lambda = (At)^{-1/b}$ into (11) gives

$$\begin{aligned}
0 &= (At)^{-c/b} \left(-\frac{c}{bt} F - \frac{a}{bt} X\partial_X F \right) - K(At)^{-(c+2a)/b}\partial_X^2 F \\
&= \frac{1}{bt}(cF + aX\partial_X F) + K(At)^{-2a/b}\partial_X^2 F \\
&= cF + aX\partial_X F + KbA^{-2a/b} t^{1-2a/b} \partial_X^2 F.
\end{aligned}$$

This time choosing $b = 2a$ removes all t-dependence from the equation; we can also remove the K by setting $A = K$. Setting $c = na$ finally leads us to

$$2F'' + XF' + nF = 0, \tag{13}$$

which is satisfied if and only if

$$(\partial_t - K\partial_x^2)(Kt)^{-n/2}\Phi(x/(Kt)^{1/2}, 1/K) = 0. \tag{14}$$

The rest of the proceedure is the same as in the first section.