Some Green's Functions

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This handout is mainly a discussion of extensions of the idea of Green's functions of Poisson's equation, with some examples. Also included are derivations of the Green's functions for the diffusion equation and the wave equation in $(1+1)$ dimensions. Sections marked with a * are not in the Methods schedule.

1 Poisson's Equation

Poisson's equation is

$$\nabla^2 u = -f. \tag{1.1}$$

Recall that a Dirichlet Green's function for Poisson's equation in a three-dimensional domain $\mathcal{D}$ is meant to satisfy the following three conditions:

1. Harmonic in x everywhere but x':

$$\nabla_x^2 G(x;x') = 0 \quad (x, x' \in \mathcal{D}, x \neq x'), \tag{1.2}$$

2. Zero on the boundary:

$$G(x;x') = 0 \quad (x \in \partial\mathcal{D}, x \in \mathcal{D}), \tag{1.3}$$

3. Local condition that causes its Laplacian to act like $\delta(x-x')$:

$$G(x'+\varepsilon;x') \sim -\frac{1}{4\pi\left|\varepsilon\right|} \tag{1.4}$$

as $|\varepsilon| \to 0$.

Globally, this allows us to write

$$G(x;x') = -\frac{1}{4\pi\left|x-x'\right|} + F(x;x'), \tag{1.5}$$

where F is harmonic in the first variable.

The other, more usual way of expressing this is through the identification

$$\nabla_x^2 G(x;x') = \delta(x-x'). \tag{1.6}$$

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1.1 *Green's Functions in n Dimensions

Taking $x' = 0$, the (hyper-)spherical symmetry implies that $G(x;0) = G(|x|)$. Integrating over a sphere of radius R around 0, we should have

$$1 = \int_{|x|<R} \nabla^2 G(r)\, dV = \int_{|x|=R} \frac{dG}{dr}(R) dS = S_{n-1} R^{n-1} G'(R) \tag{1.7}$$

using the divergence theorem and doing the integral: S_{n-1} is defined as the "surface area" of the $(n-1)$-sphere.[1] Then solving this differential equation and noting that we want $G(r) \to 0$ as $r \to \infty$, we find

$$G(x;x') = -\frac{1}{(n-2)S_{n-1}\,|x-x'|^{n-2}}. \tag{1.8}$$

Remark 1. You may wonder about the one-dimensional Green's function: it is easy to see that the only function you can take is

$$G(x;x') = \frac{1}{2}\left|x-x'\right|, \tag{1.9}$$

which has the same problem as we encounter in the two-dimensional case: namely that it does not converge to zero for large $|x|$. This causes the character of solutions to be very different: for example, this means that electric charges repel each other more as they get further apart!

1.2 The Green's Function for the Interior of a Sphere

1.2.1 An equation representing a sphere

We consider first the zero set of the function

$$x \mapsto -\frac{1}{|x-b|} + \frac{\lambda}{|x-c|}, \tag{1.10}$$

where $b \neq c$ and $\lambda > 0$; since certainly $x \neq b$, this set is expressed by the equation

$$\frac{|x-c|}{|x-b|} = \lambda. \tag{1.11}$$

Lemma 2. *If $\lambda \neq 1$, (1.11) is the equation of a sphere.*

(Note that if $\lambda = 1$ this is obviously the equation of the plane bisecting the line segment between b and c.)

[1] I.e. the integral over $|x| = r$ in n dimensions.

Proof. Squaring and rearranging, we have the following sequence of equalities:

$$\begin{aligned}
\lambda^2 |x-b|^2 &= |x-c|^2 \\
\lambda^2(|x|^2 - 2x\cdot b + |b|^2) &= |x|^2 - 2x\cdot c + |c|^2 \\
(\lambda^2-1)|x|^2 - 2x\cdot(\lambda^2 b - c) &= |c|^2 - \lambda^2|b|^2 \\
|x|^2 - 2x\cdot\frac{\lambda^2 b - c}{\lambda^2-1} &= \frac{|c|^2-\lambda^2|b|^2}{\lambda^2-1} \\
\left|x - \frac{\lambda^2 b - c}{\lambda^2-1}\right|^2 &= \frac{|c|^2-\lambda^2|b|^2}{\lambda^2-1} + \left|\frac{\lambda^2 b-c}{\lambda^2-1}\right|^2 \\
&= \frac{\lambda^2|c|^2 - 2\lambda^2 b\cdot c + \lambda^2|b|^2}{(\lambda^2-1)^2} \\
&= \frac{\lambda^2|c-b|^2}{(\lambda^2-1)^2}
\end{aligned}$$

Now, for $\lambda \neq 1$, this is essentially by definition the equation of a sphere centred at $(\lambda^2 b - c)/(\lambda^2-1)$ and with radius $\lambda|c-b|/(\lambda^2-1)$. Notice also that exactly one of b or c is inside the sphere. □

1.2.2 Three Dimensions

It is easy to see that the function in (1.10) will satisfy the conditions if we choose a scaling and appropriate values of b, c and λ. Let the sphere have radius R.

1. To satisfy Laplace's equation, we choose $b = x'$ and c outside the sphere.
2. To make the function zero on the boundary of the sphere, we need to choose c and λ so that the sphere we found in the lemma is centred at zero and has radius a. Choosing the centre is easy: set $c = \lambda^2 x'$, which reduces the equation of the sphere to

$$|x| = \lambda\left|\lambda^2 x' - x'\right|/(\lambda^2-1) = \lambda\left|x'\right|, \tag{1.12}$$

   and it follows immediately that we need to take $\lambda = R/|x'|$.
3. To obtain the locally correct form, we just scale by $1/4\pi$.

Therefore the Green's function is

$$G(x;x') = -\frac{1}{4\pi|x-x'|} + \frac{\lambda}{|x-\lambda^2 x'|}, \quad \lambda = \frac{R}{|x'|}. \tag{1.13}$$

1.2.3 *Generalisation to Higher Dimensions

It is easy to see that a similar idea generalises: if $n > 3$, the function

$$G(x;x') = -\frac{1}{(n-2)S_{n-1}|x-x'|^{n-2}} + \frac{\lambda^{n-2}}{(n-2)S_{n-1}|x-\lambda^2 x'|^{n-2}} \tag{1.14}$$

obviously still has a zero set that is the sphere of radius R, and satisfies all the other conditions to be a Green's function.

1.2.4 Two Dimensions

A similar idea works here, but since the free Green's function behaves like a logarithm, we can instead produce a function which behaves like $|x - x'|$ near $x = x'$, has no other zeros inside the sphere, and is 1 on the boundary; we then take its logarithm. However, we can simply repurpose the equation we found for a sphere: with λ as before, the set

$$\left\{ x \in \mathbb{R}^2 : \frac{\lambda |x - x'|}{|x - \lambda^2 x'|} = 1 \right\} \tag{1.15}$$

is the circle of radius R by our previous work, and hence we may try as the Green's function

$$G(x; x') = \frac{1}{2\pi} \log \left(\frac{\lambda |x - x'|}{|x - \lambda^2 x'|} \right); \tag{1.16}$$

it is easy to verify that this is satisfies the other conditions for it to be the Green's function, in that it has the correct sort of logarithmic singularity at $x = x'$ and is regular elsewhere in the sphere.

1.3 *A point of caution: compact domains without boundary

Let $\mathcal{D}$ be a compact domain without a boundary. You may well complain that you know of no such domains, but there is a very simple one: the circle, most simply parametrised as $[0, 1]$ with the ends identified (I shall come back to this specific example later). Another example is the sphere in any number of dimensions, or any closed surface, such as the torus.[2]

Suppose that u solves Poisson's equation, (1.1). If $f \equiv 0$, i.e. u is harmonic, then if we multiply Laplace's equation by u and integrate over $\mathcal{D}$, we have

$$0 = \int_{\mathcal{D}} u \nabla^2 u \, dV = 0 - \int_{\mathcal{D}} |\nabla u|^2 \, dV, \tag{1.17}$$

where the usual boundary term from Green's identity is zero since there is no boundary. But this means that u is constant, since if it is not, the right hand side is strictly negative.

If $f \not\equiv 0$, integrating (1.1) over $\mathcal{D}$, we find

$$-\int_{\mathcal{D}} f \, dV = \int_{D} \nabla^2 u \, dV = 0, \tag{1.18}$$

a contradiction unless f has total integral zero. To put this more physically, "the sum of the electric charge on a compact domain without boundary is zero".

This is a problem when we want to find the Green's function, since (working heuristically: it is easy but tedious to work rigorously here)

$$\int_{\mathcal{D}} \delta(x - x') \, dV = 1, \tag{1.19}$$

so it is not consistent to solve $\nabla^2 G = \delta$ on $\mathcal{D}$. Because $\mathcal{D}$ has finite volume $m(\mathcal{D})$,[3] there is an obvious thing to do: modify the right-hand side so that it integrates to zero, by instead solving

$$\nabla^2 G(x; x') = \delta(x - x') - \frac{1}{m(\mathcal{D})}. \tag{1.20}$$

[2] The torus presents other difficulties that can be understood using the theory of elliptic functions, which is presented in Part II Riemann Surfaces, and possibly in Part II Further Complex Methods.

[3] This is easy to see: if it was not contained in a ball of finite radius, then the open cover of $\mathcal{D}$ by the intersection of $\mathcal{D}$ with integer-radius balls, $\mathcal{D} = \bigcup_n \mathcal{D} \cap B_n(0)$, would have no finite subcover and $\mathcal{D}$ could not be compact.

This is now consistent, and considering the expression we expect to solve (1.1) we also have

$$\begin{aligned} u(x) &= \int_{\mathcal{D}} G(x;x')(-f)(x')\,dV', \\ \nabla^2 u(x) &= \int_{\mathcal{D}} \left(\nabla^2 G(x;x')\right)(-f)(x')\,dV' \\ &= \int_{D} \left(\delta(x-x') + \frac{1}{m(\mathcal{D})}\right)(-f)(x')\,dV' \\ &= -f(x) + 0, \end{aligned}$$

since $\int_{\mathcal{D}} f = 0$. Therefore (1.20) is the sensible extension of the Green's function to compact domains without boundary; it is called the *generalised Green's function*, surprisingly enough.[4]

Example For the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, we have to solve

$$G''(x;x') = \delta(x-x') - 1,$$

with $x, x' \in \mathbb{T}$. The solution is obviously of the form

$$G(x;x') = \frac{1}{2}\left|x-x'\right| - \frac{1}{2}(x-x')^2 + Ax + B;$$

it is easy to check that taking $A = B = 0$ is sufficient to make G and G_x continuous.

1.4 An Inconvenient Remark About Conventions

There are unfortunately two conventions for the Laplacian:[5] the one that you have learnt,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

and the "geometric" convention,

$$\Delta := -\nabla^2;$$

any work should state somewhere which one it is using if it uses the symbol Δ.

To see why really $(-\nabla^2)$ is a sensible thing to consider, look at the one-dimensional eigenvalue equation for the Laplacian on $[0,1]$ with zero boundary conditions, that is,

$$-u'' = \lambda u.$$

This has eigenfunctions

$$u(x) = A \sin k\pi x$$

for $k \in \{1, 2, \dots\}$; this means that with this convention, the Laplacian is a positive operator (i.e. it has positive eigenvalues), which is more conventional to think about than having negative eigenvalues.

[4] For more detail about this, see Hilbert and Courant's *Methods of Mathematical Physics*, v.1 p.355ff., where the subject is treated in more generality.

[5] You will see the same situation occur with the convention for the metric in Electrodynamics, General Relativity and Quantum Field Theory: it is especially unfortunate that GR normally uses the "mostly plus" signature $(-,+,+,+)$, while QFT normally uses "mostly minus", $(+,-,-,-)$

Another reason is that this equation can be thought of as originating from making stationary the integral

$$\frac{1}{2}\int \left(|\nabla u|^2 + \lambda u^2\right),$$

which is obviously nonnegative if $\lambda > 0$, so it has a minimum. Hence the operator $(-\nabla^2)$ is a natural one to think about in contexts where one is integrating by parts (using Green's identity) on $|\nabla u|^2$. This also explains the weird sign used in Poisson's equation.

2 Causal Green's Function for the Diffusion Equation

The forced $(1+1)$-dimensional[6] diffusion equation is

$$\theta_t - D\theta_{xx} = f(x,t); \tag{2.1}$$

let θ also satisfy the homogeneous boundary condition $\theta(0,t) = 0$.

We can find the free Green's function of the diffusion equation using the Fourier transform: since the space on which x is defined is all of $\mathbb{R}$, we can take the Fourier transform with respect to it to find

$$\tilde{\theta}_t + k^2 D\tilde{\theta} = \tilde{f}(k,t), \qquad \tilde{\theta}(k,0) = 0 \tag{2.2}$$

This equation can be solved using the integrating factor $\exp(k^2 Dt)$, and we obtain

$$\tilde{\theta}(k,t) = \tilde{\theta}(k,0)e^{-k^2Dt} + e^{-k^2Dt}\int_0^t e^{k^2D\tau}\tilde{f}(k,\tau)\,d\tau. \tag{2.3}$$

The initial condition implies that the first term is zero. The second term will require the inverse Fourier transform of $\exp(-k^2D(t-\tau))$,[7] which you should recall from the Fourier transform handout as being

$$\frac{1}{\sqrt{4\pi D(t-\tau)}}\exp\left(-\frac{x^2}{4D(t-\tau)}\right). \tag{2.4}$$

Combining this with the convolution theorem gives us

$$\theta(x,t) = \int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4D(t-\tau)}\right)f(\xi,\tau)\,d\xi\,d\tau, \tag{2.5}$$

and hence the Green's function is

$$G(x,t;\xi,\tau) = H(t-\tau)\frac{1}{\sqrt{4\pi D(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{4D(t-\tau)}\right). \tag{2.6}$$

(This is called the *causal* Green's function because the integral only depends on times that are less than t.)

[6] Physicists normally write $(1+1)$ instead of 2 when dealing with equations where one of the dimensions is treated differently: in this case, because it is time.

[7] Notice that this exists, since $t > \tau$.

2.1 The solution with a nonzero initial condition and Duhamel's principle

On the other hand, suppose we instead have an initial condition $\theta(x,0) = \Theta(x)$, say. Then we also have to deal with the first term in equation (2.3), which we do in the same way as the last term, taking the inverse Fourier transform of $\exp(-k^2 Dt)$; we find that for $t > 0$, the first term is then

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-\xi)^2}{4Dt}\right)\Theta(\xi)\,d\xi = \int_{-\infty}^{\infty} G(x,t;\xi,0)\Theta(\xi)\,d\xi. \tag{2.7}$$

The similarity of the solutions[8] (2.5) and (2.7) is the main idea behind Duhamel's principle: that having forcing in a range of times is like having a boundary condition at each time in the range. Or conversely, having a boundary condition is like having a forcing at one specific time. To put this more mathematically, the solution with initial condition $\theta(x,0) = 0$ and a forcing $f_s(x)$ at exactly one time s is

$$\theta_s(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,t;\xi,\tau) f_s(\xi)\delta(\tau - s)\,d\xi\,d\tau \tag{2.8}$$

$$= \int_{-\infty}^{\infty} G(x,t;\xi,s) f_s(\xi)\,d\xi. \tag{2.9}$$

But for $t > s$,[9] this is identical to the solution to the problem with initial condition $\theta(x,s) = f_s(x)$ and no forcing.

Conversely, suppose I have a forcing $f(x,t)$ for each time t and $\theta(x,0) = 0$. Writing $f(x,s) = f_s(x)$, the decomposition

$$f(x,t) = \int_0^{\infty} f_s(x)\delta(t-s)\,ds \tag{2.10}$$

allows us to write the solution as

$$\begin{aligned}
\theta(x,t) &= \int_0^t \int_0^{\infty} G(x,t;\xi,\tau) f(\xi,\tau)\,d\xi\,d\tau \\
&= \int_0^t \int_0^{\infty} G(x,t;\xi,\tau) \left(\int_0^{\infty} f_s(\xi)\delta(\tau-s)\,ds\right) d\xi\,d\tau \\
&= \int_0^{\infty} \left(\int_0^t \int_0^{\infty} G(x,t;\xi,\tau) f_s(\xi)\delta(\tau-s)\,d\xi\,d\tau\right) ds \\
&= \int_0^{\infty} \left(\int_0^{\infty} G(x,t;\xi,s) f_s(\xi)\,d\xi\right) ds \\
&= \int_0^t \left(\int_0^{\infty} G(x,t;\xi,s) f_s(\xi)\,d\xi\right) ds,
\end{aligned}$$

which we recognise as a sum[10] of solutions to unforced problems with initial conditions $\theta(x,t) = f_s(x)$ for $0 \leqslant s \leqslant t$.

Note If you have been paying attention, you might object that all I've done is change how I write f and relabel τ to s. This is not quite all that I have done: the point of doing so is that τ is thought of as a variable time (like t) over which we are integrating, whereas s is a fixed parameter that labels a

[8] Not a pun. Move along, please.

[9] Remember G contains a Heaviside step function!

[10] or rather, integral

particular function f_s. Meanwhile, $f(x,\tau)$ is a function of two variables, but $f_s(x)$ is a collection of functions of x only, parametrised by s.

So, essentially the idea of Duhamel's principle is that both of these points of view:

- (the forcing $f(x,\tau)$, time τ), and
- (the initial conditions $f_s(x)$, parameter s)

lead to the same results.

3 Causal Green's Function for the Wave Equation

We can do a similar thing for the forced wave equation in $(1+1)$ dimensions,

$$y_{tt} - c^2 y_{xx} = f(x,t), \tag{3.1}$$

with the initial conditions $y(x,0) = y_t(x,0) = 0$. Fourier transforming with respect to x gives

$$\tilde{y}_{tt} + c^2k^2\tilde{y} = \tilde{f}(k,t), \qquad \tilde{y}(k,0) = \tilde{y}_t(k,0) = 0 \tag{3.2}$$

We recognise this as a forced SHM-type equation. You should have calculated the Green's function for this initial value problem before: it is

$$g(t;\tau) = H(t-\tau)\frac{\sin kc(t-\tau)}{kc}, \tag{3.3}$$

so the solution in Fourier space is

$$\tilde{y}(x,t) = \int_0^t \frac{\sin kc(t-\tau)}{kc}\tilde{f}(k,\tau)\,d\tau; \tag{3.4}$$

we shall have the solution in real space using the convolution theorem, once we apply the inverse Fourier transform to $(kc)^{-1}\sin kc(t-\tau)$. The simplest way to do this is to note that

$$\int_{-c(t-\tau)}^{c(t-\tau)} \frac{1}{2c}e^{ikx}\,dx = \frac{\sin kc(t-\tau)}{kc}, \tag{3.5}$$

as we wanted, so the function is $1/2c$ for $|x| < c(t-\tau)$ and 0 elsewhere;[11] this can also be written as

$$\frac{1}{2c}H(c(t-\tau) - |x|). \tag{3.6}$$

It follows immediately using the convolution theorem that the solution in real space is

$$y(x,t) = \frac{1}{2c}\int_0^t \int_{-\infty}^{\infty} H(c(t-\tau) - |x-\xi|)f(\xi,\tau)\,d\xi\,d\tau \tag{3.7}$$

$$= \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\,d\tau \tag{3.8}$$

(the easiest way to see this is to consider the regions where the integrand is not zero). Therefore the Green's function is

$$G(x,t;\xi,\tau) = \frac{1}{2c}H\left(c(t-\tau) - |x-\xi|\right). \tag{3.9}$$

Exactly the same argument we gave in the previous section for the diffusion equation gives a Duhamel's principle for the wave equation.

[11] Of course, the Fourier inversion theorem would say it takes the value $1/4c$ at the edges, but these are single points and so the values do not affect the Fourier transform.