The Riemann Zeta Function at Positive Even Integers

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This is a sequel of sorts to the handout deriving an unusual type of series for the cotangent, *viz.*

$$\pi \cot \pi z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}. \tag{0.1}$$

1 Bernoulli numbers

The *Bernoulli numbers* are defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.1}$$

It is easy to find that the first few are

$$B_0 = 1, \qquad B_1 = -\frac{1}{2}, \qquad B_2 = \frac{1}{6}, \qquad B_3 = 0, \qquad B_4 = -\frac{1}{30}, \qquad B_5 = 0, \qquad \dots$$

In particular, for $n > 0$, $B_{2n+1} = 0$. This is easy to see by considering the odd/even splitting of the generating function: the odd part is

$$\frac{1}{2}\left(\frac{t}{e^t - 1} - \frac{-t}{e^{-t} - 1}\right) = \frac{1}{2}\left(\frac{t}{e^t - 1} - \frac{te^t}{e^t - 1}\right) = -\tfrac{1}{2}t.$$

On the other hand, for us the interesting part is the even part:

$$\frac{1}{2}\left(\frac{t}{e^t - 1} + \frac{-t}{e^{-t} - 1}\right) = \frac{1}{2}\left(\frac{t}{e^t - 1} + \frac{te^t}{e^t - 1}\right) = \frac{t}{2}\frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}} = \tfrac{1}{2}t \coth \tfrac{1}{2}t. \tag{1.2}$$

This immediately gives us:

2 Three Trigonometric Power Series

We discover the power series of the hyperbolic cotangent: changing variables, we find

$$\coth z = \frac{1}{z} + \sum_{n=1}^{\infty} B_{2n} 2^{2n} \frac{z^{2n-1}}{(2n)!}, \tag{2.1}$$

one of the trigonometrical Taylor expansions we cannot find easily. Now, $\cot z = i \coth iz$ so we also get the Taylor expansion of the cotangent for free:

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1}, \tag{2.2}$$

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[It is easy to see using the double-angle formulae that

$$\tan z = \cot z - 2\cot 2z, \tag{2.3}$$

and so we get the tangent series almost for free as well:

$$\tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} z^{2n-1}. \tag{2.4}$$

(Obviously the hyperbolic tangent is the same, sans the $(-1)^{n-1}$.) Using

$$\cot \tfrac{1}{2}z - \cot z = \operatorname{cosec} z \tag{2.5}$$

gives the cosecant series,

$$\operatorname{cosec} z = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2(2^{2n-1}-1)B_{2n}}{(2n)!} z^{2n-1}. \tag{2.6}$$

Secant is far less considerate, unfortunately: it has its own set of numbers, also known as the Euler numbers.[1]]

3 The Riemann Zeta Function

The *Riemann zeta function* is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1; \tag{3.1}$$

its analytic continuation is easily the most important function in analytic number theory. Euler famously argued that $\zeta(2) = \pi^2/6$ using the sine product formula (discussed on the previous sheet). We shall proceed using the two equivalent expressions for the cotangent we just derived.

We have, using the Binomial Theorem,

$$\frac{2z^2}{z^2 - n^2} = -\frac{2z^2/n^2}{1 - z^2/n^2} = \sum_{k=1}^{\infty} -2\frac{z^{2k}}{n^{2k}}. \tag{3.2}$$

Then, the series (0.1) becomes

$$\pi z \cot \pi z = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} -2\frac{z^{2k}}{n^{2k}} = 1 + \sum_{k=1}^{\infty} -2z^{2k}\zeta(2k), \tag{3.3}$$

swapping the order of summation.[2] On the other hand, changing variables in the series (2.2) produces

$$\pi z \cot \pi z = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi)^{2k} B_{2k}}{(2k)!} z^{2k}, \tag{3.4}$$

and equating coefficients gives, for $k \geqslant 1$,

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}B_{2k}}{2(2k)!} \tag{3.5}$$

[1]Not to be confused with Euler's other number. Or Euler's other other number. Or Euler's other$(1 - \text{other})^{-1}$ number

[2]Exercise: show why this is allowed for $|z|$ sufficiently small.