Adding Angular Momentum

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1 Example Sheet 3, Question 6

This is a nasty question: the calculations are unpleasent and any errors will tend to propagate like the proverbial rabbits. Let's go carefully. Sadly to find the states listed in the question, we have to calculate the first three rows of the multiplet:

$$\begin{array}{llll} |4\,4\rangle & & & \\ |4\,3\rangle & |3\,3\rangle & & \\ |4\,2\rangle & |3\,2\rangle & |2\,2\rangle & \\ & & |2\,1\rangle & (|1\,1\rangle?) \end{array} \tag{1}$$

$|4\,4\rangle$ This is obviously $|3\,3\rangle\,|1\,1\rangle$; we have no choice in the matter.

$|4\,3\rangle$ Apply J_- to $|4\,4\rangle$ in two different ways:

$$\begin{aligned} J_-\,|4\,4\rangle &= \sqrt{(4+4)(4-4+1)}\,|4\,3\rangle = \sqrt{8}\,|4\,3\rangle \\ &= (J_-\,|3\,3\rangle)\,|1\,1\rangle + |3\,3\rangle\,(J_-\,|1\,1\rangle) \\ &= \sqrt{(3+3)(3-3+1)}\,|3\,2\rangle\,|1\,1\rangle + \sqrt{(1+1)(1-1+1)}\,|3\,3\rangle\,|1\,0\rangle\,; \end{aligned}$$

simplifying gives

$$|4\,3\rangle = \frac{\sqrt{3}}{2}\,|3\,2\rangle\,|1\,1\rangle + \frac{1}{2}\,|3\,3\rangle\,|1\,0\rangle\,. \tag{2}$$

$|3\,3\rangle$ We need a state orthogonal to $|4\,3\rangle$, made from the same product states. The obvious choice is

$$|3\,3\rangle = \frac{1}{2}\,|3\,2\rangle\,|1\,1\rangle - \frac{\sqrt{3}}{2}\,|3\,3\rangle\,|1\,0\rangle \tag{3}$$

$|3\,2\rangle$ Now it gets unpleasant. I'm going to stop writing out the non-simplified forms of the coefficients to save space. We act with J_- on $|3\,3\rangle$:

$$\begin{aligned} J_-\,|3\,3\rangle &= \sqrt{6}\,|3\,2\rangle \\ &= \frac{1}{2}\left(\sqrt{10}\,|3\,1\rangle\,|1\,1\rangle + \sqrt{2}\,|3\,2\rangle\,|1\,0\rangle\right) - \frac{\sqrt{3}}{2}\left(\sqrt{6}\,|3\,2\rangle\,|1\,0\rangle + \sqrt{2}\,|3\,3\rangle\,|1\,{-1}\rangle\right), \end{aligned}$$

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which simplifies to

$$|3\,2\rangle = \frac{1}{2}\sqrt{\frac{5}{3}}\,|3\,1\rangle\,|1\,1\rangle - \frac{1}{\sqrt{3}}\,|3\,2\rangle\,|1\,0\rangle - \frac{1}{2}\,|3\,3\rangle\,|1\,{-1}\rangle\,. \tag{4}$$

$|2\,2\rangle$ We actually can't do this yet: we need $|4\,2\rangle$ so that we can find a vector orthogonal to both of these.

$$\begin{aligned} J_-\,|4\,3\rangle &= \sqrt{14}\,|4\,2\rangle \\ &= \frac{\sqrt{3}}{2}\left(\sqrt{10}\,|3\,1\rangle\,|1\,1\rangle + \sqrt{2}\,|3\,2\rangle\,|1\,0\rangle\right) + \frac{1}{2}\left(\sqrt{6}\,|3\,2\rangle\,|1\,0\rangle + \sqrt{2}\,|3\,3\rangle\,|1\,{-1}\rangle\right), \end{aligned}$$

so

$$|4\,2\rangle = \frac{1}{2}\sqrt{\frac{15}{7}}\,|3\,1\rangle\,|1\,1\rangle + \sqrt{\frac{3}{7}}\,|3\,2\rangle\,|1\,0\rangle + \frac{1}{2\sqrt{7}}\,|3\,3\rangle\,|1\,{-1}\rangle\,. \tag{5}$$

Now, we have to find a state with $M = 2$ that is orthogonal to both $|4\,2\rangle$ and $|3\,2\rangle$, which must then be proportional to $|2\,2\rangle$. Solving the equations

$$\begin{aligned} \frac{1}{2}\sqrt{\frac{5}{3}}a - \frac{1}{\sqrt{3}}b - \frac{1}{2}c &= 0 \\ \frac{1}{2}\sqrt{\frac{15}{7}}a + \sqrt{\frac{3}{7}}b + \frac{1}{2\sqrt{7}}c &= 0 \end{aligned}$$

gives

$$|2\,2\rangle = \frac{1}{\sqrt{21}}\,|3\,1\rangle\,|1\,1\rangle - \sqrt{\frac{5}{21}}\,|3\,2\rangle\,|1\,0\rangle + \sqrt{\frac{5}{7}}\,|3\,3\rangle\,|1\,{-1}\rangle\,. \tag{6}$$

$|2\,1\rangle$ Nearly there. Applying J_- to $|2\,2\rangle$:

$$\begin{aligned} J_-\,|2\,2\rangle &= 2\,|2\,1\rangle \\ &= \frac{1}{\sqrt{21}}\left(2\sqrt{6}\,|3\,0\rangle\,|1\,1\rangle + \sqrt{2}\,|3\,1\rangle\,|1\,0\rangle\right) - \sqrt{\frac{5}{21}}\left(\sqrt{10}\,|3\,1\rangle\,|1\,0\rangle + \sqrt{2}\,|3\,2\rangle\,|1\,{-1}\rangle\right) \\ &\quad + \sqrt{\frac{5}{7}}\left(\sqrt{6}\,|3\,2\rangle\,|1\,{-1}\rangle\right), \end{aligned}$$

or on rearranging,

$$|2\,1\rangle = \frac{1}{\sqrt{7}}\,|3\,0\rangle\,|1\,1\rangle - 2\sqrt{\frac{2}{21}}\,|3\,1\rangle\,|1\,0\rangle + \sqrt{\frac{10}{21}}\,|3\,2\rangle\,|1\,{-1}\rangle\,. \tag{7}$$

Now we have to try to calculate $|1\,1\rangle$. Explicit calculation shows that the only possible states in the row with $M = 1$ are

$$|3\,0\rangle\,|1\,1\rangle\,, |3\,1\rangle\,|1\,0\rangle\,, |3\,2\rangle\,|1\,{-1}\rangle\,, \tag{8}$$

so the space is three-dimensional; it is easy to check that the combined states $|4\,1\rangle\,, |3\,1\rangle\,, |2\,1\rangle$ are by construction orthonormal, and hence they span the three-dimensional space with $M = 1$. We are supposed to find $|1\,1\rangle$ by choosing a vector orthogonal to these three, but the only vector which has vanishing inner products with all elements of a basis is the zero vector.

Alternatively, we can take a general linear combination of (8) and apply J_+: imposing that $J_+ |1\,1\rangle = 0$ then implies that all of the coefficients must be zero. We shall give a quick summary of this way, since it is in general *much* quicker if you're careful.

Explicitly, suppose

$$|1\,1\rangle = a\,|3\,0\rangle\,|1\,1\rangle + b\,|3\,1\rangle\,|1\,0\rangle + c\,|3\,2\rangle\,|1\,{-1}\rangle\,. \tag{9}$$

Now, J_+ acts on $|j\,m\rangle$ by

$$J_+\,|j\,m\rangle = \sqrt{(j-m)(j+m+1)}\,|j\,m\rangle\,, \tag{10}$$

so obviously we have $J_+ |1\,1\rangle = 0$.[1] Applying it to the other form,

$$0 = a\left(\sqrt{12}\,|3\,1\rangle\,|1\,1\rangle + 0\right) + b\left(\sqrt{10}\,|3\,2\rangle\,|1\,0\rangle + \sqrt{2}\,|3\,1\rangle\,|1\,1\rangle\right) + c\left(\sqrt{6}\,|3\,3\rangle\,|1\,{-1}\rangle + \sqrt{2}\,|3\,2\rangle\,|1\,0\rangle\right), \tag{11}$$

giving the equations

$$\begin{aligned} \sqrt{12}a + \sqrt{2}b &= 0 \\ \sqrt{10}b + \sqrt{2}c &= 0 \\ \sqrt{6}c &= 0, \end{aligned} \tag{12}$$

implying that all three of a, b, c are zero, so the entire state is zero. The orthogonalisation proof is similar, but contains considerably more calculation.

2 Example Sheet 3, Question 7

We shall do this without calculating all the states: instead, let's use a symmetry and counting argument. Let J_- be an angular momentum lowering operator, and the particles each have total angular momentum J. We call the particle-switching operator P.

Obviously $|2J\,2J\rangle$, the highest-weight top state, is symmetric under switching the particles, since it can only consist of one state: $|J\,J\rangle\,|J\,J\rangle$.

Now, what about the states $|2J\,M\rangle$? J_- acts on product states by $J_- = j_- \otimes 1 + 1 \otimes j_-$, where j_- is the lowering operator on each factor of the product.[2] The obvious symmetry in the definitions means that $[J_-, P] = 0$, and hence *the lowering operator preserves (anti)symmetry.* (We can also see this directly from the definition.) Therefore, since the top state is symmetric, we find that the entire "column" of $4J+1$ states is symmetric.

Now, since $[J^2, P] = 0$ and $[J_3, P] = 0$, and there is only one state with each pair of eigenvalues (i.e. $\{J^2, J_3\}$ is a complete commuting set of operators), then *every* state $|j\,m\rangle$ in the whole spin multiplet has a definite symmetry, because it must also be an eigenstate of P.

The next part is more difficult: we have to show that the $m = 2J - k$ multiplet has (anti)symmetric states if k is even (odd). All we have to do is find the symmetry of the top state $|2J-k\;\;2J-k\rangle$. To do this, consider the form of the states: the previous paragraph shows that each state in the multiplet is made out of a set of eigenstates of P of the form

$$|p; q\rangle_\pm := \frac{1}{\sqrt{2}}\left(|J\,p\rangle\,|J\,q\rangle \pm |J\,q\rangle\,|J\,p\rangle\right); \tag{13}$$

[1] One needs to be very careful here to remember to do the calculation with J_+, having done a few dozen with the slightly different J_-!

[2] Remember, this is actually the definition of the action of the momentum operators on the tensor product.

in particular, we have $-m \leqslant p, q \leqslant m$ and $p+q=m$. This will be a basis of the same space as the $m = 2J-k$ "row", and this row has $k+1$ elements,

$$|2J \;\; m\rangle\,, |2J-1 \;\; m\rangle\,, |2J-2 \;\; m\rangle\,, \ldots, |2J-k \;\; m\rangle\,. \tag{14}$$

There are always $\lceil k/2 \rceil$ of the $|p;q\rangle_+$, and there are always $\lfloor k/2 \rfloor$ of the $|p;q\rangle_-$ (think about what happens to the states with $p=q$ in the odd case; the rest should be fairly clear).

Now, each state in the row has a fixed parity, and they are all meant to be orthogonal. Clearly any symmetric state is orthogonal to any antisymmetric one; the span of the row splits into a direct sum of the symmetric and antisymmetric states with $m = 2J-k$. Now we apply a counting argument to find out where the state $|2J-k \;\; 2J-k\rangle$ goes. We proceed by induction.

Basis cases We need separate basis cases for odd and even k. If $k=0$, then we just have the highest-weight state $|2J \;\; 2J\rangle$, which we observed before was symmetric. Now, the next one along is $k=1$, which is a state in the two-dimensional space with $m = 2J-1$, which also contains the symmetric $|2J \;\; 2J-1\rangle$. Hence to be orthogonal to this, and therefore antisymmetric, as you already know from the many calculations of this sort you have done.

Inductive step Assume that the first $k-1$ columns are alternately symmetric and antisymmetric. Suppose first that k is even. Then the subspace of symmetric states is already spanned by the $k/2$ states $|2J \;\; 2J-k\rangle\,, \ldots, |2J-k+1 \;\; 2J-k\rangle$, so the new state $|2J-k \;\; 2J-k\rangle$ must be in the antisymmetric subspace by orthogonality. The odd case works in exactly the same way, except it is now the antisymmetric subspace that is already full.

From this, it follows in general that *the states $|j\,m\rangle$ of the tensor product of two identical angular momentum algebrae are symmetric when $2J-j$ is even and antisymmetric when $2J-j$ is odd.* In particular, the first part of the question now follows.

Moreover, this also tells us the part about $L+S$ being even: we need the whole state to be even, so it either lives in $\mathrm{Ang}^A \otimes \mathrm{Sp}^A$ or $\mathrm{Ang}^S \otimes \mathrm{Sp}^S$. The above result shows that we need L and S to be either both odd or both even, and hence either way the sum is even.

Lastly, suppose that $J=1$. Then we have from the usual angular momentum algebra that

$$|L-S| \leqslant 1 \leqslant L+S; \tag{15}$$

since $L-S$ has to be even as well, we must therefore have $L=S$. Moreover S is 0, 1 or 2, and we need $L+S \geqslant 1$, so the only possibilities are $L=1, S=1$ or $L=2, S=2$.