Interactions and Conservation Laws

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All interactions are assumed to preserve angular momentum.[1] In this course we also assume that parity is conserved, although this is not always true (see last section).

Reminder The total angular momentum operator is $\mathbf{J} = \mathbf{L} + \mathbf{S}$. By addition of angular momenta,

$$J \in \{|L-S|, |L-S|+1, \dots, L+S-1, L+S\} \tag{1}$$

(If one of L or S is zero, however, J has only one possible value.)

1 General proceedure

Given interaction

$$\sum_i X_i \to \sum_j Y_j, \tag{2}$$

1. Write down given properties of the particles.
2. Specify that the orbital angular momentum L of the left-hand side is zero (we can do this by choosing the reference frame appropriately).
3. Find the possible values of the total angular momentum $J = L + S$ on the right-hand side.
4. If possible, use any symmetries of the Y_j (e.g., if they are identical, can use spin-statistics and swap operations).
5. Use parity to further narrow down the options:

   $$\prod_i \eta_{X_i} = (-1)^L \prod_j \eta_{Y_j} \tag{3}$$

   Note that parity only acts on the orbital angular momentum.

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[1] If they don't, there's probably a particle missing that you haven't found in your detector.

2 Example Sheet 4, Question 3

Suppose X has spin 1 and Y has spin 0. We are considering the interaction

$$X \to Y + Y. \tag{4}$$

Consider the left-hand side. We assume that $L = 0$, so the total angular momentum is $J = L + S = 0 + 1 = 1$. Therefore the total angular momentum on the right-hand side is $J = 1$. Now, the total spin on the right can only be 0, because that is the only possible value to make from two spin-0 orbitals. Hence $L = 1$ on the right.

But this is impossible: since $Y + Y$ are two identical particles with spin 0, the wavefunction must be invariant under swapping them, so the parity is positive. But the parity of a system with $L = 1$ is negative. Hence this decay cannot occur.

For the second part, we have the data:

Particle	Spin	η
ρ	1	?
$\pi^{\pm}, \pi^0$	0	-1

(5)

$$\rho^+ \to \pi^+ + \pi^0 \tag{6}$$

We impose $L = 0$ on the left again. Then $J = S = 1$ on the left-hand side. By the same argument as above, the right-hand side has $L = 1$ and $S = 0$. Now use parity:

$$(-1)^0 \eta_\rho = (\eta_{\pi^+})(\eta_{\pi^0}) \times (-1). \tag{7}$$

Hence $\eta_\rho = -1$.

Notice that the initial state has definite parity, which implies that the final state does as well. And since the particles are different, this is not in contradiction with the first part of the question.

3 Example Sheet 4, Question 4

Now we have a lot more data to start with:

Particle	Spin	η
ρ	1	-1
$\pi^{\pm}$	0	-1
X	?	?
K	0	?

(8)

$$X \to \rho^+ + \pi^- \tag{9}$$

$$X \to K + K \tag{10}$$

Start with $X \to K + K$. We do an argument similar to Q3: look first at the right-hand side.

1. Total spin of $K + K$ is $S = 0$, from adding angular momenta.[2]
2. Two identical spin-0 particles $\implies$ wavefunction is symmetric under interchange.
3. $S = 0 \implies$ spin part of wavefunction is symmetric $\implies$ spatial part is symmetric.

[2]Hence if K had spin 1, the total spin could be 0, 1, or 2.

4. But spatial part behaves as

$$\psi(-x) = (-1)^L (\eta_K)^2 \psi(x) = (-1)^L \psi(x). \tag{11}$$

Hence L must be even, and hence $J = L + 0$ is even.

Therefore, J is even for the right-hand side. Conservation of momentum implies J is even on the left, and since we set $L = 0$ on the left, we have $S = J$ even for X, i.e., X has even spin.

Next, we have parity conservation:

$$\eta_X = (-1)^L (\eta_K)^2 = 1, \tag{12}$$

again since L is even. Hence we know the parity of X.

Now consider the other decay. From the data we have, the total spin on the right-hand side is 1. We now have to examine the possibilities for X's spin, and find the smallest which does not produce contraditions. From what we know so far, we have the following possibilities for low spin:

Spin of X	Possible values of L for $\rho^+ + \pi^-$
0	1
2	$1, 2, 3$
$\vdots$	$\vdots$

(13)

On the left, the spin of X is the same as the total angular momentum since $L = 0$. On the right, we have $L \in \{S-1, S, S+1\}$ provided that $S > 0$.

- If the spin of X is 0, then J for X is 0. Conservation of momentum gives $J = 0$ for $\rho^+ + \pi^-$ as well.

  Since $S = 1$ on the right-hand side, we must have $L = 1$.[3] But this is inconsistent with parity: if L is odd on the right-hand side, then $\eta_X = \eta_\pi \eta_\rho (-1)^L = -1$, which contradicts the parity of X we found from $X \to K + K$. Hence we cannot have $S = 0$ for X.

- If the spin of X is 2, then $J = 2$ on the right.

  $S = 1$ on right $\implies$ $L \in \{1, 2, 3\}$. By the parity argument in the previous point, L has to be even. Ah, but now we have a possible even L!

Therefore, the lowest possible spin that X can have is 2. We also found $\eta_X = 1$.

3.1 Summary

1. Set $L = 0$ for X.

2. Look at $X \to K + K$.

   K has spin 0 $\implies$ total spin $S = 0$ for $K + K$.

   But 2 identical spin-0 particles $\implies$ wavefunction is symmetric, and since $\psi(-x) = (\eta_K)^2(-1)^L\psi(x)$, L is even for $K + K$.

   $\implies$ J is even.

3. By conservation of angular momentum, J is even for X. Since $L = 0$, $J = S$ for X.

   By conservation of parity, $\eta_X = (\eta_K)^2(-1)^L = 1$.

[3] If $L = 0$, $J = 1$, if $L = 2$, $J \in \{1, 2, 3\}$, and so on.

4. Now look at $X \to \rho^+ + \pi^-$. ρ^+ has spin 1, π^- has spin 0.

   $\implies$ total spin for $\rho^+ + \pi^-$ is $S = 1$.

5. Now we consider possible values of spin for X.

   $S = 0$ ($\implies J = 0$) Conservation of angular momentum $\implies J = 0$ for $\rho^+ + \pi^-$, but $S = 1 \implies L = 1$.

   However, conservation of parity $\implies 1 = \eta_X = (\eta_\rho)(\eta_\pi)(-1)^L \implies L$ is even #

   $S = 2$ ($\implies J = 2$) $J = 2$ for $\rho^+ + \pi^- \implies L \in \{1, 2, 3\}$.

   L is even $\implies L = 2$, therefore $S = 2$ is possible.

6. X can have spin 2, $\eta_X = 1$. This is the lowest possible spin for X.

4 A note about parity and weak interactions

The strong interaction preserves parity. However, the weak interaction does not: processes such as the familiar $d \to u + e^- + \nu_e$ (one form of beta decay) cannot be analysed in the same way as the above.

The Standard Model manifestation of this is that the neutrino field does not couple to the weak interaction field in the same way as other fermions: the weak interaction is *chiral*, and we only observe left-handed neutrinos. This is fine if the neutrino is massless.[4]

However, recently, the three neutrino flavours have been observed to oscillate over time. This is only possible if neutrinos have mass.

Technical explanation Here I shall give an appallingly quick summary of how fermions are described relativistically. The first equation Schrödinger wrote down to describe quantum particles was the (relativistically consistent) Klein–Gordon equation,

$$(\partial_0^2 - \nabla^2 + m^2)\psi = 0, \tag{14}$$

but in Fourier form, the equation

$$-E^2 + p^2 + m^2 = 0 \tag{15}$$

has solutions with negative energy, which were regarded as unphysical, so he took a low-energy limit to obtain the equation that bears his name. Dirac had the insight to take a "square root" of the operator $\partial_0^2 - \nabla^2$, at the expense of having to use matrices and a new type of object, a *spinor*, to represent the function, instead of the scalars used up to that point.[5] His equation is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{16}$$

which is appallingly beautiful. γ^μ are a set of four matrices, satisfying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, which can actually be made out of the Pauli matrices:

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \tag{17}$$

(this is not the only representation, but it is the easiest to use to see what is going on chirally). We can then split the four-dimensional spinor ψ into two two-dimensional ones, $\psi = (\psi_L, \psi_R)$, the *left-handed* and *right-handed* parts: these are precisely the chiral components of the particle.

Now, in this form the Dirac equation splits into

$$i\sigma^\mu \partial_\mu \psi_L = m\psi_R, \quad i\bar{\sigma}^\mu \partial_\mu \psi_R = \psi_L \tag{18}$$

(where $\sigma^\mu = (I_2, \sigma^k)$ and $\bar{\sigma}^\mu = (I_2, -\sigma^k)$). If (and only if) the mass is zero, the right-hand sides are zero, and the chiral components do not mix.

Okay, so that's why fermions with mass are achiral, and fermions without mass can be chiral. Only left-handed neutrinos have been observed to date, so only the left-handed neutrino field is included in the Standard Model. Since neutrinos obey the Dirac equation, this is only possible if they are massless. Neutrinos only couple to the weak interaction, so the weak interaction does not preserve parity.

So what's the problem with just having massless neutrinos?

Neutrino oscillations are explained by the mass eigenstates of neutrinos having different eigenvalues, but also differing from the flavour eigenstates (i.e., whether they are ν_e, ν_μ, or ν_τ). Neutrinos are

[4] But still travels at the speed limit, despite what some experimenters may tell you.

[5] A spinor is similar to a vector, but is only rotated by half the angle that physical space is wher acted on by a rotation.

always produced with definite flavour, and hence in this model, begin in a certain mixture of the mass eigenstates. Each mass eigenstate then propagates at a different speed, so once you measure the neutrino later on, it looks like a different combination of flavour eigenstates. Oscillations between all three flavours have been observed, so for this theory you need three different masses, and hence at least two of the mass eigenstates must have nonzero eigenvalues.