Some important probability distributions

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1 Discrete

Specialisations are indicated by ↳ . Distributions you *need* to know about are in **bold**. (My convention is $\mathbb{N} = \{0, 1, 2, \dots\}$.)

Distribution	Notation	Support	Parameters	Mass function $\mathbb{P}(X = r)$	$\mathbb{E}[X]$	$\operatorname{Var} X$	$\mathbb{E}[t^X]$
Uniform	$\operatorname{Unif}\{a, b\}$	$\{a, a+1, \dots, b\}$	$a, b \in \mathbb{Z},\ a < b$	$\frac{1}{b-a+1}$	$\frac{a+b}{2}$	$\frac{(b-a+2)(b-a)}{12}$	$\frac{t^a - t^{b+1}}{n(1-t)}$
Binomial	$\mathrm{B}(n, p)$	$\{0, 1, \dots, n\}$	$n \in \mathbb{N},\ p \in [0, 1]$	$\binom{n}{r} p^r (1-p)^r$	np	$np(1-p)$	$(1-p+pt)^n$
↳ **Bernoulli**[1]	$\operatorname{Ber}(p)$	$\{0, 1\}$	$p \in [0, 1]$	$p^r (1-p)^{1-r}$	p	$p(1-p)$	$(1-p) + pt$
Negative binomial	$\operatorname{NegBin}(k, p)$	$\{k, k+1, \dots\}$	$k \in \mathbb{N},\ p \in [0, 1]$	$\binom{r-1}{k-1} p^r (1-p)^{r-k}$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2}$	$\frac{(pt)^k}{(1-(1-p)t)^k}$
↳ **Geometric**[2]	$\operatorname{Geo}(p)$	$\mathbb{N}$	$p \in [0, 1]$	$p(1-p)^{r-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pt}{1-(1-p)t}$
Poisson	$\operatorname{Po}(\lambda)$	$\mathbb{N}$	$\lambda \in [0, \infty)$	$\frac{e^{-\lambda} \lambda^r}{r!}$	λ	λ	$e^{\lambda(t-1)}$
Hypergeometric[3]	$\operatorname{Hypergeo}(N, k, n)$	$\{\max\{0, n+K-N\}, \dots, \min\{n, K\}\}$	$N \in \mathbb{N}$, $K, n \in \{0, 1, \dots, N\}$	$\binom{K}{r}\binom{N-K}{n-r} \Big/ \binom{N}{n}$	$\frac{nK}{N}$	$\frac{nK(N-K)(N-n)}{N^2(N-1)}$	$\frac{\binom{N-K}{n}}{\binom{N}{n}} {}_2F_1(-n, -K; N-K-n+1; t)$
Beta-binomial[4]	$\operatorname{BetaBin}(n, \alpha, \beta)$	$\{0, 1, \dots, n\}$	$n \in \mathbb{N},\ \alpha, \beta \in (0, \infty)$	$\binom{n}{k} \frac{B(k+\alpha, n-k+\beta)}{B(\alpha, \beta)}$	$\frac{n\alpha}{\alpha+\beta}$	$\frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$\frac{{}_2F_1(-n; \alpha; 1-\beta-n; t)}{{}_2F_1(-n; \alpha; 1-\beta-n; 1)}$
Multinomial	$\operatorname{Multi}(n, p_1, \dots, p_k)$	$(r_1, \dots, r_n) \in \{0, 1, \dots, n\}^k$, $\sum_{i=1}^k r_i = n$	$p_i \in [0, 1], \sum_{i=1}^k p_i = 1$	$\frac{n!}{r_1! \cdots r_k!} p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$	$(np_1, \dots, np_k)$	$\operatorname{cov}(X_i, X_j) = \begin{cases} np_i(1-p_i) & i = j \\ -np_i p_j & i \neq j \end{cases}$	$\mathbb{E}[t_1^{X_1} \cdots t_k^{X_k}] = \left(\sum_{i=1}^k p_i t_i\right)^n$

[1] $\operatorname{Ber}(p) = \mathrm{B}(1, p)$ [2] $\operatorname{Geo}(p) = \operatorname{NegBin}(1, p)$ [3] ${}_2F_1(a, b; c; z) = \sum_{n=0}^\infty \left(\prod_{k=0}^{n-1} \frac{(a+k)(b+k)}{c+k}\right) \frac{z^n}{n!}$ is the *hypergeometric function*. [4] $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx$ is the *Beta-function*.

Stable sums Suppose that X and Y are independent. Then:

$X \sim$	$Y \sim$	$X + Y \sim$
$\mathrm{B}(m, p)$	$\mathrm{B}(n, p)$	$\mathrm{B}(m+n, p)$
$\operatorname{NegBin}(k, p)$	$\operatorname{NegBin}(l, p)$	$\operatorname{NegBin}(k+l, p)$
$\operatorname{Po}(\lambda)$	$\operatorname{Po}(\mu)$	$\operatorname{Po}(\lambda+\mu)$

Such a family, where the sum of two gives a third with different parameters, is called *stable*.

Other relationships

- $X \sim \mathrm{B}(n, p) \implies n - X \sim \mathrm{B}(n, 1-p)$.
- If $X \sim \operatorname{Geo}(1-q)$, $Y \sim \operatorname{Geo}(1-q')$ are independent, $\min\{X, Y\} \sim \operatorname{Geo}(1-qq')$.

2 Continuous

Distribution	Notation	Support	Parameters	Density $f(x)$	$\mathbb{E}[X]$	$\operatorname{Var} X$	$\mathbb{E}[e^{itX}]$
Uniform	$\mathrm{Unif}[a,b]$	$[a,b]$	$a,b\in\mathbb{R},\ a<b$	$\dfrac{1}{b-a}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$	$\dfrac{e^{ibt}-e^{iat}}{it(b-a)}$
Normal	$\mathrm{N}(\mu,\sigma^2)$	$\mathbb{R}$	$\mu\in\mathbb{R},\ \sigma\in(0,\infty)$	$\dfrac{1}{\sigma\sqrt{2\pi}}\exp\left(-\dfrac{(x-\mu)}{2\sigma^2}\right)$	μ	σ^2	$e^{it\mu-t^2\sigma^2/2}$
Gamma[1]	$\mathrm{Gamma}(\alpha,\lambda)$	$[0,\infty)$	$\alpha\in(0,\infty),\ \lambda\in(0,\infty)$	$\dfrac{\lambda^\alpha x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}$	$\dfrac{\alpha}{\lambda}$	$\dfrac{\alpha}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-it}\right)^\alpha$
↳ **Exponential**[2]	$\mathrm{Exp}(\lambda)$	$[0,\infty)$	$\lambda\in(0,\infty)$	$\lambda e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\dfrac{\lambda}{\lambda-it}$
↳ Chi-squared[3]	χ^2_k	$(0,\infty)$	$k\in\{1,2,\dots\}$	$\dfrac{1}{2^{k/2}\Gamma(k/2)}x^{k/2-1}e^{-x/2}$	k	$2k$	$\dfrac{1}{(1-2it)^{k/2}}$
Beta[4]	$\mathrm{Beta}(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in(0,\infty)$	$\dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$	$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	${}_1F_1(\alpha;\alpha+\beta;it)$
Student's t [5]	T_ν	$\mathbb{R}$	$\nu\in(0,\infty)$	$\dfrac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)}\dfrac{1}{(1+x^2/\nu)^{(\nu+1)/2}}$	$\begin{cases}0 & \nu>1\\ - & \nu\leqslant 1\end{cases}$	$\begin{cases}\nu/(\nu-2) & \nu>2\\ \infty & 1<\nu\leqslant 2\\ - & \nu\leqslant 1\end{cases}$	$\dfrac{(\sqrt{\nu}\lvert t\rvert)^{\nu/2}K_{\nu/2}(\sqrt{\nu}\lvert t\rvert)}{2^{\nu/2-1}\Gamma(\nu/2)}$
↳ Cauchy[6]	$\mathrm{Cauchy}(m,\gamma)$	$\mathbb{R}$	$m\in\mathbb{R},\ \gamma>0$	$\dfrac{\gamma}{\pi\big((x-m)^2+\gamma^2\big)}$	$-$	∞ [7]	$e^{itm-\gamma\lvert t\rvert}$
Multivariate normal	$\mathrm{N}_d(\mu,\Sigma)$	$\mathbb{R}^d$	$\mu\in\mathbb{R}^d,\ \Sigma\in\mathbb{R}^{d\times d}_{>0}$	$\dfrac{\exp\big(-(x-\mu)^T\Sigma^{-1}(x-\mu)/2\big)}{(2\pi)^{d/2}\sqrt{\det\Sigma}}$	μ	$\operatorname{cov}(X_i,X_j)=\Sigma_{ij}$	$\mathbb{E}[e^{it^TX}]=e^{it^T\mu+t^T\Sigma t/2}$

[1] $\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}\,dx$, the *Gamma-function.* [2] $\mathrm{Exp}(\lambda)=\mathrm{Gamma}(1,\lambda)$ [3] $\chi^2_k=\mathrm{Gamma}(k/2,1/2)$ [4] ${}_1F_1(a;b;z)=\sum_{n=0}^\infty\left(\prod_{k=0}^{n-1}\frac{a+k}{b+k}\right)\frac{z^n}{n!}$ is the *confluent hypergeometric function.* [For B see overleaf.] [5] $K_\nu(x)$ is a modified Bessel function. [6] $\mathrm{Cauchy}(0,1)=T_1$ [7] In the sense that $V(x)=\mathbb{E}[(X-x)^2]$, which normally has the variance as its minimum, has no finite values.

Stable sums Suppose that X and Y are independent. Then:

$X\sim$	$Y\sim$	$X+Y\sim$
$\mathrm{N}(\mu,\sigma^2)$	$\mathrm{N}(\nu,\tau^2)$	$\mathrm{N}(\mu+\nu,\sigma^2+\tau^2)$
$\mathrm{Gamma}(\alpha,\lambda)$	$\mathrm{Gamma}(\beta,\lambda)$	$\mathrm{Gamma}(\alpha+\beta,\lambda)$
$\mathrm{Cauchy}(m,\gamma)$	$\mathrm{Cauchy}(n,\delta)$	$\mathrm{Cauchy}(m+n,\gamma+\delta)$

Other relationships If $a\neq 0, b\in\mathbb{R}$,

- $X\sim\mathrm{N}(\mu,\sigma^2)\implies aX+b\sim\mathrm{N}(a\mu+b,a^2\sigma^2)$
- If $X\sim\mathrm{N}(\mu,\sigma^2)$, $Y\sim\mathrm{N}(\nu,\tau^2)$ are independent, then $aX+bY\sim\mathrm{N}(a\mu+b\nu,a^2\sigma^2+b^2\tau^2)$
- $X\sim\mathrm{Gamma}(\alpha,\lambda)\implies aX\sim\mathrm{Gamma}(\alpha,a\lambda)$
- $X\sim\mathrm{Cauchy}(m,\gamma)\Rightarrow aX+b\sim\mathrm{Cauchy}(am+b,\lvert a\rvert\gamma)$
- $X\sim\mathrm{Cauchy}(0,\gamma)\implies 1/X\sim\mathrm{Cauchy}(0,1/\gamma)$
- $X\sim\mathrm{Beta}(\alpha,\beta)\implies 1-X\sim\mathrm{Beta}(\beta,\alpha)$
- If $X\sim\mathrm{Exp}(\lambda)$, $Y\sim\mathrm{Exp}(\mu)$ are independent, $\min\{X,Y\}\sim\mathrm{Exp}(\lambda+\mu)$.
- $X\sim\mathrm{Exp}(\lambda)\implies\lfloor X\rfloor\sim\mathrm{Geo}(1-e^{-\lambda})$.
- If $X_i\sim\mathrm{N}(0,1)$ are independent, $\sum_{i=1}^k X_i^2\sim\chi^2_k$.
- If $X,Y\sim\mathrm{N}(0,1)$ are independent, $X/Y\sim\mathrm{Cauchy}(0,1)$.
- If $X\sim\mathrm{N}(0,1)$ and $U\sim\chi^2_k$, then $X/\sqrt{U/k}\sim T_k$.
- If $X\sim\mathrm{Gamma}(\alpha,\lambda)$ and $Y\sim\mathrm{Gamma}(\beta,\lambda)$, are independent, then $X/(X+Y)\sim\mathrm{Beta}(\alpha,\beta)$ (and $X/(X+Y)$ and $X+Y$ are independent).