# Gambler's Ruin the easy way

**With Martingales**

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## 1 Gambler's Ruin

Let $(X_k)$ be a sequence of IID random variables taking value 1 with probability $p$ and $-1$ with probability $q = 1 - p$, and define $S_n = \sum_{k=1}^{n} X_k$. We consider the interval $\{a, a+1, \dots, b\}$, where $a < b$ are integers. This is the classical *gambler's ruin* problem. Define

$$T = \min\{n : S_n \in \{a, b\}\},$$

the time until $S_n$ becomes equal to $a$ or $b$. Notice that either $S_T = a$ or $S_T = b$.

We are interested in quantities like $\mathbb{P}(S_T = b)$ (i.e. the probability that a particular player wins) and $\mathbb{E}[T]$, the expected playing time.

$S_T$ is a sum of a random number of independent random variables, but since $T$ is not independent of the $X_k$ (think about small cases), we cannot apply the usual formula for expectations.

In IA Probability, you learn how to find these quantities using recurrence relations, conditioning on the first step of the random walk. In this handout, we are going to discuss a much quicker way to extract these, and more general information, from the problem, using a new concept.

## 2 Martingales

**Definition 1.** A sequence $(M_k)_{k=0}^{\infty}$ of random variables with $\mathbb{E}[|M_k|] < \infty$ is called a *martingale*[1] if

$$(\forall n \geqslant 0) \quad \mathbb{E}[M_{n+1} \mid M_0, \dots, M_n] = M_n.$$

That is, given what's happened so far, tomorrow you expect to be in the same place you are today.

Here are some examples of martingales:

1. Let $(Z_k)_{k=1}^{\infty}$ be a sequence of independent random variables with $\mathbb{E}[Z_k] = 0$. Then $(Z_k)$ is a martingale since $\mathbb{E}[Z_{k+1} \mid Z_0, \dots, Z_k] = 0 = \mathbb{E}[Z_k]$.
2. With the same $Z_k$ as above, define $Y_n = \sum_{k=1}^{\infty} Z_k$. Then

$$\mathbb{E}[Y_{n+1} \mid Y_1, \dots, Y_n] = \mathbb{E}[Y_n + Z_{n+1} \mid Y_1, \dots, Y_n] = Y_n + \mathbb{E}[Z_{n+1}] = Y_n,$$

since $\mathbb{E}[Z_{n+1}] = 0$ and $\mathbb{E}[Y_n \mid Y_n] = Y_n$.

3. Suppose that the $Z_k$ are as above but with common variance $\sigma^2$. Then $W_n = Y_n^2 - n\sigma^2$ is a martingale.

[1] The origins of this name are obscure: it appears to have originally been a French type of harness, then became the name of a particular betting system, before being adopted for this.

4. More generally, given $Z_k$ a sequence of independent random variables with $\mathbb{E}[Z_k] = \mu$ and $\operatorname{Var}(Z_k) = \sigma^2$, and setting $Y_k = \sum_{k=1}^n Z_k$, then
$$M_n^{(1)} = Y_n - n\mu, \qquad M_n^{(2)} = (Y_n - n\mu)^2 - n\sigma^2$$
are both martingales.

5. It looks like the previous two martingales might be two terms in a sequence of martingales. Indeed this is the case: suppose that $Z_k$ has characteristic function $\varphi_k(t)$. Then $U_n = \exp(itS_n)/\prod_{k=1}^n \varphi_k(t)^n$ is a martingale. (Notice that $U_{n+1} = U_n \exp(itZ_{n+1})/\varphi_{n+1}(t)$, and then the result follows by "taking out what is known".)

   Expanding as a series in $t$ about $t = 0$ and using $\varphi_k(0) = 1$ gives
$$1+it(S_n-\sum_{k=1}^n \varphi'(0))-\frac{t^2}{2}\left(\left(S_n - \sum_{k=1}^n \varphi'(0)\right)^2 - \sum_{k=1}^n(\varphi''(0) - \varphi'(0)^2)\right)+O(t^3) = 1+itM_n^{(1)}-\frac{t^2}{2}M_n^{(2)}+O(t^3),$$
   explaining the previous martingales.

6. If the $Z_k$ have finite MGFs $m_k(\theta)$, the same idea shows that $V_n = \exp(\theta S_n)/\prod_{k=1}^n m_k(\theta)^n$ is a martingale.

While the latter martingale does not always exist, it has several advantages when it does, since it is a real-valued logarithmically convex function.[2] We will exploit this in the next section.

There is plenty more to be said about martingales, in continuous time too, (see, for example, II Stochastic Financial Models, or III Advanced Probability) but we confine ourselves to the simplest case, which we use as a demonstration of their power.

## 2.1 Stopping times

**Definition 2.** Let $(X_n)$ be a stochastic process (that is, a sequence of random variables). Let $T$ be a random variable taking values in $\mathbb{N} \cup \{\infty\}$. We say that $T$ is a *stopping time* for $(X_n)$ if
$$\mathbb{E}[\mathbb{1}\{T \leqslant n\} \mid X_1, \dots, X_n] = \mathbb{1}\{T \leqslant n\}.$$

That is, if we know the values of $X_1, \dots, X_n$, we know whether $T \leqslant n$. Another way to think about this, using the gambling interpretation of probability: a stopping time is a strategy for choosing when to stop gambling. [3] For example: "When I'm out of money", "When I've doubled my original stake", "When it's time for lunch" and so on. It should be easy to interpret the most of following examples in this way:

1. $T = a$: a constant is a stopping time.
2. More generally, if $T$ is independent of $X_n$, $T$ is a stopping time.
3. If $S$ is a set, $T = \min\{n : X_n \in S\}$ is a stopping time, called the *hitting time* of $S$.
4. If $S, T$ are stopping times, then so are $S + T$, $\min\{S, T\}$ and $\max\{S, T\}$.
5. In contrast, $\max\{n : X_n \in S\}$ is not usually a stopping time.

In particular, in Gambler's Ruin the time to absorption is a stopping time.

[2] Rather unsurprisingly, a function $f$ is *logarithmically convex* (or log-convex for short) if $\log f$ is a convex function.
[3] Or for when to sell your shares, same thing.

### 2.2 The magic part: the Optional Stopping Theorem

The last thing about martingale theory we will discuss is the following remarkable theorem.

**Theorem 3** (Doob's Optional Stopping Theorem). *Let $(M_n)$ be a martingale with $|M_{n+1} - M_n|$ bounded, and $T$ be a stopping time for $(M_n)$ with $\mathbb{P}(T < \infty) = 1$. Then*

$$\mathbb{E}[M_T] = \mathbb{E}[M_0].$$

This is not even the most general form of this theorem, and nor is this the most powerful theorem in martingale theory. But it suffices for what we need.

## 3 Application to Gambler's Ruin

Gambler's Ruin is special, because it has only two absorbing states. This means that we need very little information to determine what it does, essentially because

$$\mathbb{E}[f(T)g(S_T)] = g(b)\mathbb{E}[f(T)\mathbb{1}\{S_T = b\}] + g(a)\mathbb{E}[f(T)\mathbb{1}\{S_T = a\}],$$

so all that is required is to calculate these two expectations. In particular, knowing two quantities involving the same function is normally enought to determine the expectation of the function.

### 3.1 Hitting probability

We start with $\mathbb{P}(S_T = a)$. The trick that we shall exploit time and again is the following: $X_k$ are IID and bounded, so the MGF $m(\theta) = \mathbb{E}[e^{\theta X_1}]$ exists and $V_n(\theta) = \exp(\theta S_n)m(\theta)^{-n}$ exists and is a martingale.

Suppose first that $\mathbb{E}[X_1] \neq 0$. Then $m'(\theta) \neq 0$, and $m$ is log-convex and so convex. This implies that $m(\theta) = 1$ has exactly 2 real roots: 0, and some other number $\zeta$ which has the opposite sign to $\mathbb{E}[X_1]$. Thus $\exp(\zeta S_n)$ is a martingale. (The other root gives the constant martingale 1.)

We apply the Optional Stopping Theorem to this martingale, giving

$$\mathbb{E}[\exp(\zeta S_T) \mid S_0 = x] = \mathbb{E}[\exp(\zeta S_0) \mid S_0 = x] = e^{\zeta x}.$$

But, provided we know that $\mathbb{P}(S_T = a \mid S_0 = x) + \mathbb{P}(S_T = b \mid S_0 = x) = 1$, this is enough information, because the above gives the additional equation

$$\begin{aligned} e^{\zeta x} &= \mathbb{E}[\exp(\zeta S_T)\mathbb{1}\{S_T = a\} \mid S_0 = x] + \mathbb{E}[\exp(\zeta S_T)\mathbb{1}\{S_T = b\} \mid S_0 = x] \\ &= \mathbb{E}[\exp(\zeta a)\mathbb{1}\{S_T = a\} \mid S_0 = x] + \mathbb{E}[\exp(\zeta b)\mathbb{1}\{S_T = b\} \mid S_0 = x] \\ &= e^{\zeta a}\mathbb{P}(S_T = a \mid S_0 = x) + e^{\zeta b}\mathbb{P}(S_T = b \mid S_0 = x), \end{aligned}$$

and now we can solve these two simultaneous equations to find

$$\begin{aligned} \mathbb{P}(S_T = a \mid S_0 = x) &= \frac{e^{\zeta b} - e^{\zeta x}}{e^{\zeta b} - e^{\zeta a}} \\ \mathbb{P}(S_T = b \mid S_0 = x) &= \frac{e^{\zeta x} - e^{\zeta a}}{e^{\zeta b} - e^{\zeta a}}, \end{aligned}$$

avoiding recurrence relations completely.

## 3.2 Expected time to absorption/game length

We consider the martingale $M_n^{(1)} = S_n - n\mu$. The Optional Stopping Theorem implies that

$$\mathbb{E}[S_T - T\mu \mid S_0 = x] = \mathbb{E}[S_0 - 0\mu \mid S_0 = x] = x,$$

so we immediately see that

$$\begin{aligned}
\mu\mathbb{E}[T \mid S_0 = x] &= -x + \mathbb{E}[S_T \mid S_0 = x] \\
&= -x + a\mathbb{P}(S_T = a \mid S_0 = x) + b\mathbb{P}(S_T = b \mid S_0 = x) \\
&= -x + a\frac{e^{\zeta b} - e^{\zeta x}}{e^{\zeta b} - e^{\zeta a}} + b\frac{e^{\zeta x} - e^{\zeta a}}{e^{\zeta b} - e^{\zeta a}} \\
&= \frac{(a-x)e^{\zeta b} + (b-a)e^{\zeta x} + (x-b)e^{\zeta a}}{e^{\zeta b} - e^{\zeta a}}
\end{aligned}$$

## 3.3 All the information desirable: the joint MGF

This is all well and good, but ideally we would like to derive all the information about these distributions at once. The method remains similar to our approach in previous cases, however.

We would like to know $\mathbb{E}[e^{\phi T + \theta S_T} \mid S_0 = x]$. Notice that $\exp(S_n\theta + n\phi)$ is a martingale if $\phi = -\log(m(\theta))$, where $m$ is the MGF of $X_k$. We now suppress the $S_0 = x$ condition for brevity. Then by the Optional Stopping Theorem,

$$e^{\theta x} = \mathbb{E}[e^{\theta S_T + \phi T}] = e^{\theta a}\mathbb{E}[e^{\phi T}\mathbb{1}\{S_T = a\}] + e^{\theta b}\mathbb{E}[e^{\phi T}\mathbb{1}\{S_T = b\}].$$

Since $m$ is convex and tends to $+\infty$ as $\theta \to \pm\infty$ (provided that $X_i$ takes both positive and negative values with positive probability), for every $\phi < 0$ there are exactly two solutions to $m(\theta) = e^{\phi}$, call these $\theta_\pm$. Then the above equation becomes two,

$$\begin{aligned}
e^{\theta_+ x} &= e^{\theta_+ a}\mathbb{E}[e^{\phi T}\mathbb{1}\{S_T = a\}] + e^{\theta_+ b}\mathbb{E}[e^{\phi T}\mathbb{1}\{S_T = b\}] \\
e^{\theta_- x} &= e^{\theta_- a}\mathbb{E}[e^{\phi T}\mathbb{1}\{S_T = a\}] + e^{\theta_- b}\mathbb{E}[e^{\phi T}\mathbb{1}\{S_T = b\}],
\end{aligned}$$

and solving these equations gives

$$\mathbb{E}[e^{\phi T}\mathbb{1}\{S_T = a\}] = \frac{e^{\theta_- b + \theta_+ x} - e^{\theta_+ b + \theta_- x}}{e^{\theta_- b + \theta_+ a} - e^{\theta_+ b + \theta_- a}} \qquad \mathbb{E}[e^{\phi T}\mathbb{1}\{S_T = b\}] = \frac{e^{\theta_- x + \theta_+ a} - e^{\theta_+ x + \theta_- a}}{e^{\theta_- b + \theta_+ a} - e^{\theta_+ b + \theta_- a}}$$

from which we obtain the MGF

$$\mathbb{E}[e^{\phi T + \theta S_T} \mid S_0 = x] = \frac{e^{\theta a}(e^{\theta_- b + \theta_+ x} - e^{\theta_+ b + \theta_- x}) + e^{\theta b}(e^{\theta_- x + \theta_+ a} - e^{\theta_+ x + \theta_- a})}{e^{\theta_- b + \theta_+ a} - e^{\theta_+ b + \theta_- a}}.$$

This can now be used to calculate essentially anything we want, by differentiating enough and putting $\phi, \theta = 0$.

*Remark* 4. Significantly, *nothing we have done here assumes the random walk is simple.* Everything still applies if, for example, we have a nonzero probability of staying in the same place, or, with careful modification of the definition of $S_T$, if we can take steps in units of larger than one: the MGF's convexity is all we need for the analysis. This is a vast improvement over the recurrence relation analysis, where even the slightest adjustment changes everything!

*Remark* 5. The above analysis has assumed $\mu \neq 0$, in which case it is easy to check that $\mathbb{E}[T] < \infty$ (the random walk drifts towards one of the barriers on average). When $\mu = 0$, provided that $\mathbb{E}[T]$ remains finite, we can take the limit to obtain results: remember that since $m'(0) = \mu = 0$, the origin is the unique minimum of the MGF, and $\sigma > 0$, so the roots have expansions of the form $\theta_\pm(\phi) = \pm\sqrt{2\phi}/\sigma + O(\phi)$.

On the other hand, the case $\mu = 0$ often actually makes things *easier* to calculate: the list of Wald martingales then begins

$$M_n^{(1)} = S_n \qquad M_n^{(2)} = S_n^2 - \sigma^2 n \qquad M_n^{(3)} = S_n^3 - 3\sigma^2 n S_n - n\mu_3,$$

where $\mu_3 = \mathbb{E}[X^3]$. From these it is possible to calculate $\mathbb{E}[T^p S_T^q]$ inductively for any $p, q$, and for small values this is easier than the MGF martingale method due to the complexity of the root expressions in general.

## 4 Wald identities

We are familiar with

**Theorem 6.** *Let $(X_k)_{k=1}^{\infty}$ be independent with finite mean $\mu$ and set $S_n = \sum_{k=1}^{n} X_k$. Let $N$ be a positive integer-valued random variable, independent of the $X_k$, with $\mathbb{E}[N] < \infty$. Then*

$$\mathbb{E}[S_N] = \mu \mathbb{E}[N]. \tag{1}$$

*If in addition $\operatorname{Var}(X_k) = \sigma^2$ and $\mathbb{E}[N^2]$ are finite,*

$$\operatorname{Var}(S_N) = \mu^2 \operatorname{Var}(N) + \sigma^2 \mathbb{E}[N]. \tag{2}$$

Theorems relating the expectation of sums of a random number of random variables to the expectation of their parts are called *Wald identities*.

The above result does not apply to $S_T$ from the previous section, since $T$ is not independent of the $X_k$. However, $T$ is a stopping time, and so the Optional Stopping Theorem[4] enables us to prove some similar identities:

**Theorem 7** (Wald identities for stopping times). *Let $(X_k)_{k=1}^{\infty}$ be independent with finite mean $\mu$ and set $S_n = \sum_{k=1}^{n} X_k$. Let $T$ be a stopping time for the $X_k$, with $\mathbb{E}[T] < \infty$. Then*

$$\mathbb{E}[S_T] = \mu \mathbb{E}[T]. \tag{3}$$

*If in addition $\operatorname{Var}(X_k) = \sigma^2$ and $\mathbb{E}[T^2]$ are finite,*

$$\mathbb{E}[(S_T - T\mu)^2] = \sigma^2 \mathbb{E}[T] \tag{4}$$

*Lastly, if the $X_k$ are IID with finite MGF $m(\theta)$,*

$$\mathbb{E}[\exp(\theta S_T)(m(\theta))^{-T}] = 1. \tag{5}$$

One proves this by applying the Optional Stopping Theorem to the now-familiar martingales $M_n^{(1)}$, $M_n^{(2)}$ and $V_n$. The identities (3) and (5) are familiar, but (4) contains a surprise: expanding, we find that

$$\begin{aligned}
\sigma^2 \mathbb{E}[T] &= \mathbb{E}[(S_T - T\mu)^2] \\
&= \mathbb{E}[S_T^2] - 2\mu \mathbb{E}[T S_T] + \mu^2 \mathbb{E}[T^2] \\
&= \mathbb{E}[S_T^2] - \mathbb{E}[S_T]^2 + \mathbb{E}[S_T]^2 - 2\mu \mathbb{E}[T S_T] + \mu^2 \mathbb{E}[T^2] - \mu^2 \mathbb{E}[T]^2 + \mu^2 \mathbb{E}[T]^2 \\
&= \operatorname{Var}(S_T) - 2\mu \operatorname{cov}(T, S_T) + \mu^2 \operatorname{Var}(T),
\end{aligned}$$

using the first Wald identity (3) to rewrite some of the $\mathbb{E}[S_T]$s and $\mathbb{E}[T]$s. Hence

$$\operatorname{Var}(S_T) = \sigma^2 \mathbb{E}[T] - \mu^2 \operatorname{Var}(T) + 2\mu \operatorname{cov}(T, S_T).$$

Now, if $T$ is independent of the $X_k$, we find by the Law of Total Expectation that

$$\mathbb{E}[T S_T] = \mathbb{E}[\mathbb{E}[T S_T \mid T]] = \mathbb{E}[T \mathbb{E}[S_T \mid T]] = \mathbb{E}[T \mu T] = \mu \mathbb{E}[T^2],$$

and so in this case $\operatorname{cov}(T, S_T) = \mu \operatorname{Var}(T)$, and we recover the formula (2).

But in the Gambler's Ruin case, it turns out that if $b - a$ is even and $x = (b+a)/2$, we can calculate using the joint MGF that $\operatorname{cov}(T, S_T) = 0$, and so we find, surprisingly,

$$\operatorname{Var}(S_T) = \sigma^2 \mathbb{E}[T] - \mu^2 \operatorname{Var}(T),$$

with the *opposite* sign to the expected[5] one!

[4] Although probably a more general version than the version we gave above.
[5] Pun not intended.