Short Note About The Riemann Zeta Function

Primes, the Explicit Formula, zeros, and the Riemann Hypothesis

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This note is intended to be a slightly more proper version of the summary I gave at the end of Probability Sheet 2, it is not meant to be in any way complete or authoritative!

1 Definition

In his famous paper,[1] Riemann uses

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \operatorname{Re} s > 1 \tag{1}$$

extensively. This function was originally studied by Euler, who produced many of your favourite identities like $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ and so on, but it was Riemann that really demonstrated how deeply connected it is to the primes, and proved most of its important properties, so it seems reasonable to name it after him.

2 Euler Product

One result that Euler derived (and that the question gets you to derive in a very different way) is the following product

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \qquad \operatorname{Re} s > 1; \tag{2}$$

this is called the *Euler product* of the zeta function.

3 Analytic continuation

The series (1) and product (2) that we currently have only work for $\operatorname{Re} s > 1$. We would really like to have a function defined on as much of the complex plane as possible. It turns out there is a unique way of doing this that maintains that $\zeta(s)$ is complex-differentiable ("analytic"). Riemann writes down an integral over a contour in the complex plane that can be shown to be equal to the series in $\operatorname{Re} s > 1$, but exists as an analytic function with a finite value for all s with one exception: at $s = 1$ it looks like $1/(s-1)$.

4 Functional equation

The analytically continued ζ-function (which we still call ζ since this version supersedes our original definition) has the following functional equation, which Riemann also proves (in two different ways) in his paper:

$$\zeta(1-s) = 2(2\pi)^{-s} \cos(\tfrac{1}{2}\pi s)\Gamma(s)\zeta(s). \tag{3}$$

In particular, this tells us about the region $\operatorname{Re} s < 0$ in terms of the region $\operatorname{Re} s > 1$, where we know quite a lot. It also suggests that the line $\operatorname{Re} s = 1/2$ is significant, since it is mapped to itself.

[1] "On the Number of Primes Less Than a Given Magnitude", original "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse", *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, Nov. 1859. This paper, one of the most famous in mathematics, essentially founded the field of analytic number theory. Using s for the argument also originates in this paper, a useful convention at least partly because it helps distinguish it from Weierstrass's function $\zeta(z)$, a pseudo-elliptic function that is the integral of $\wp(z)$.

5 The number of primes less than a given magnitude

Probably the most extraordinary claim in Riemann's paper is the following formula, called *Riemann's explicit formula*:

$$\Pi(x) = \mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) - \log 2 + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\log t} \tag{4}$$

(this is not Riemann's notation, his is rather more generic), where:

- The function on the left is

  $$\Pi(x) = \sum_{n=1}^{\infty} \frac{1}{n}\pi(x^{1/n})$$

  and $\pi(x)$ is the number of primes less than x. For a given x this always terminates since eventually $x^{1/n} < 2$. For large x the first term dominates significantly: using the Prime Number Theorem (see below) gives that it is asymptotically $x/\log x$, while the next term is asymptotically $x^{1/2}/\log x$.

- $\mathrm{li}(x) = \int_0^x dt/\log t$ is the logarithmic integral, which had been observed by Gauss to be a good approximation for $\pi(x)$ from looking at tables of both functions; Riemann's paper was intended as a large generalisation of this paper. This is the "main term"; it comes from the simple pole of $\zeta(s)$ at $s = 1$.

- ρ is a sum over the zeros of $\zeta(s)$ in the strip $0 \leqslant \operatorname{Re} s \leqslant 1$.

- The zeros with $\operatorname{Re} s < 0$ contribute the last term, which is tiny: even for $x = 2$ its value is < 0.15.

The magnitude of the contributions in the second-most-significant term are determined by the real parts of the zeros, more on this in the next section.

The formula (4) was derived by Riemann in a way that even at the time was regarded as somewhat suspect, and as with many of Riemann's extraordinary contributions, required some time to be put on a firmer footing; in this case it was done by von Mangoldt in 1895, who proved a variant based on a different prime-counting function, and then converted it into Riemann's own formula.

6 Zeros

Because an infinite product of complex numbers cannot converge to 0 unless one of the factors is 0, Euler's product (2) tells us that $\zeta(s)$ has no zeros with $\operatorname{Re} s > 1$.

The functional equation tells us that the only zeros in $\operatorname{Re} s < 0$ are at negative even integers: these are called *trivial zeros*, firstly because they were so easy to find, secondly because their contribution to the explicit formula (4) is uninterestingly small. Both of these were known to Riemann.

Therefore all the interesting zeros of $\zeta(s)$ lie in the strip $0 \leqslant \operatorname{Re} s \leqslant 1$.

In 1896, Hadamard and de la Vallée Poussin independently proved the *Prime Number Theorem*,

$$\pi(x) \sim \frac{x}{\log x},$$

by showing that $\zeta(s)$ has no zeros with $\operatorname{Re} s = 1$. Hence we are left with the *critical strip*, $0 < \operatorname{Re} s < 1$ (notice that this means that the "main term" in the explicit formula (4) really does dominate each of the terms in the sum over the zeros).

The functional equation also tells us that there is symmetry in the zeros: a zero at $1/2 - s$ in the critical strip implies that there is one at $1/2 + s$. The zeta function has real coefficients and so $\overline{\zeta(\bar{s})} = \zeta(s)$, which means that the zeros are also symmetric about the real axis.

Recalling that the real parts of the zeros affect the magnitude of the corrective terms in the Explicit Formula (4), and that if we have off the critical line, the previous paragraph implies we will have at least one with real part $> 1/2$, the best possible scenario for the size of the correction terms is that all the zeros lie on the *critical line* $\operatorname{Re} s = 1/2$; this is the *Riemann hypothesis*, now the most famous unsolved problem in Mathematics.

Although the result remains open, we have various partial results and evidence:

- The Prime Number Theorem restricts us to looking in the open strip instead of the closed one. In fact a refinement of this gives a *zero-free region*: de la Vallée Poussin showed in 1899 there are no zeros with $\operatorname{Re} s \geqslant 1 - C/\log|\Im s|)$ for some positive constant C, and this has been improved over the years by various people, the Wikipedia article on the Riemann hypothesis[2] has a list of recent papers and results in this area.
- G.H. Hardy proved in 1915 that there are infinitely many zeros on the critical line. This is interesting but tells us very little else: not only could there be infinitely many off the critical line, but their density could be such that the proportion of zeros on the critical line is 0 (to put this on a firm footing, imagine a rectangle of the critical strip of finite height, and count those on and off in it).
- The best current result is that, by this measure, about 5/12 of the zeros are on the critical line; this was proven very recently by Pratt, Robles, Zaharescu, and Zeindler.[3]
- As far as numerical calculations go, the evidence is pretty overwhelming: the most recent paper, by [4] calculated the positions of the first $12 \cdot 10^{12}$ zeros, up to imaginary part roughly $3 \cdot 10^{12}$. (Obviously all were on the critical line, or you would have heard about the counterexample!)
- The ideas in the zeta function has been extended in lots of other situations for other series with terms resembling n^{-s}, leading to many other zeta functions (and their cousins with other arithmetic functions in the numerator, L-functions), many of which have most of the same characteristics as Riemann's zeta function, including equivalents of Euler products, functional equations, the explicit formula, and of course their own equivalent of the Riemann hypothesis. Some of these equivalent RHs are proven.

7 Further Reading

1. Wikipedia contributors, "On the Number of Primes Less Than a Given Magnitude," Wikipedia, The Free Encyclopedia, https://en.wikipedia.org/w/index.php?title=On_the_Number_of_Primes_Less_Than_a_Given_Magnitude&oldid=1187516111 (accessed February 23, 2024).

   The Wikipedia article on Riemann's paper is pretty solid, containing a description of the entire contents of the paper, and links to the original and a translation

2. Edwards, H. (2001) *Riemann's Zeta Function*, Dover.

   A good starting point for the theory, covers from Riemann to Hardy and some of the newer results about the zeros. Quite approachable, and reasonably-priced since it's a Dover book.

3. Titchmarsh, E. C. (1930), *The Theory of the Riemann Zeta-function*, Second edition OUP 1987, revised with more up-to-date information by D. R. Heath-Brown.

   A much more technical book that, goes into a lot of the details of almost every aspect of $\zeta(s)$, including much of the material surrounding the Riemann hypothesis, at least as known in 1930. (Heath-Brown updated this substantially, but obviously there's only so much that can be done while maintaining the integrity of a book this age.)

4. Zagier, D. (1977) 'The First 50 Million Prime Numbers'. *The Mathematical Intelligencer* **1** (Suppl 2), 7–19 https://doi.org/10.1007/BF03351556

   A nice readable popular(ish) article about topics around the explicit formula. Contains graphs of the first few terms in the explicit formula. [Paywalled, but you can read it from eduroam or other university places.]

5. Garrett, P. (2010) 'Riemann's Explicit/Exact formula' https://web.archive.org/web/20230221132422/https://www-users.cse.umn.edu/~garrett/m/mfms/notes_c/mfms_notes_02.pdf

   Expositary article about the explicit formula that goes into rather more detail about the formula, including a sketch of a derivation. [Original still available at https://www-users.cse.umn.edu/~garrett/m/mfms/notes_c/mfms_notes_02.pdf, but Internet Archive link given to ideally counteract link rot.]

[2] https://en.wikipedia.org/wiki/Riemann_hypothesis#Zero-free_regions
[3] Pratt, Robles, Zaharescu, and Zeindler (2018), 'More than five-twelfths of the zeros of ζ are on the critical line', preprint available at https://arxiv.org/abs/1802.10521.
[4] Platt and Trudigan, (2021), 'The Riemann hypothesis is true up to $3 \cdot 10^{12}$', preprint available at https://arxiv.org/abs/2004.09765