Non-Degeneracy in One Dimension

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In one dimension, given certain conditions on the potential, it is actually possible to prove that each eigenenergy has precisely one associated normalisable eigenstate. Thus the point of this handout (this, of course, is a substantial part of ES1, Q8) is to prove

Theorem 1 (Non-degeneracy of energy in one dimension). *Suppose that V is bounded. Then if ψ_1 and ψ_2 satisfy*

$$\int_{-\infty}^{\infty} |\psi_i|^2 < \infty, \tag{1}$$

(the definition of normalisable) and

$$\psi_i'' = (V - E)\psi_i, \tag{2}$$

up to numerical constants (i.e. they satisfy the Schrödinger equation). Then $\psi_1 = c\psi_2$ for some $c \in \mathbb{C}$.

Since both satisfy the same differential equation, it seems sensible when talking about uniqueness to consider the Wronskian, $W = \psi_1\psi_2' - \psi_2\psi_1'$. We have

$$W' = \psi_1\psi_2'' - \psi_2\psi_1'' = 0,$$

so W is a constant. It now suffices to show (recall IA Differential Equations) that $W = 0$: this is a sufficient condition providing that two solutions to a linear differential equation are linearly dependent. How are we going to do this? The only places where we expect to know something are $x \to \pm\infty$. Our first instinct is that the normalisability allows us to say immediately that $\psi_i \to 0$, but this is actually false in general, so we also need to use the differential equation. (Easy exercise: construct some annoying counterexamples.)

One thing we can do is take the absolute value squared of both sides of the equation and integrate it: this gives

$$\int_a^b |\psi_i''|^2 = \int_a^b (V - E)^2 |\psi|^2 .$$

Since V is bounded, $V - E$ is bounded, by M, say. Then the right-hand side is smaller than $M \int_{-\infty}^{\infty} |\psi|^2$ and hence the left-hand side is bounded.

At this point we have to use two results that you probably haven't seen before, and one you may have. I'm just going to quote them: two are simple consequences of material in Analysis II, the other is a fiddly inequality.

Proposition 2. *Suppose that f'' exists, $\int_0^{\infty} |f|^2, \int_0^{\infty} |f''|^2 < \infty$. Then*

$$\left(\int_0^{\infty} |f'|^2\right)^2 \leqslant 4\int_0^{\infty} |f|^2 \int_0^{\infty} |f''|^2$$

(One can find this, with proof, in Hardy, Littlewood and Pólya, *Inequalities*, § 7.8.)

Proposition 3. *If $\int_{-\infty}^{\infty} |f'|^2 < \infty$, then f is uniformly continuous on $\mathbb{R}$.*

Proposition 4. *If $\int_{-\infty}^{\infty} |f|^2 < \infty$ and f is uniformly continuous on $\mathbb{R}$, $f(x) \to 0$ as $x \to \infty$.*

The first of these, applied twice, tells us that $\int_{-\infty}^{\infty} |\psi'|^2 < \infty$. The second we can apply twice, to $f = \psi'$ and $f = \psi$, to find that ψ and ψ' are both uniformly continuous on $\mathbb{R}$, and then the third implies that both $\psi(x)$ and $\psi'(x) \to 0$ as $x \to \pm\infty$, and hence we find immediately that $W = 0$.

(One may find a different proof of this result, which works for more general V, in Messiah, *Quantum Mechanics*, Ch. 3, § 9.)

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