Summary of Concepts in Vector Calculus

Richard Chapling v1 14 August 2017

Many concepts in this course work in n dimensions, but we stick to 2 and 3 for definiteness.

Notation We use summation convention unless indicated otherwise.

- We denote vectors in $\mathbb{R}^3$ by bold, $\mathbf{x}$. The Euclidean norm is then a function $\mathbb{R}^3 \to \mathbb{R}$ given by $\|\mathbf{x}\| = (\sum_i x_i x_i)^{1/2}$.
- We write L for a generic curve, S for a generic surface, and V for a generic volume, all of which we assume are closed sets.
- We write ∂A for the boundary of the set A: for example, if V is $\{\mathbf{x} : r \leqslant \|\mathbf{x}\| \leqslant R\}$, then $\partial V = \{\mathbf{x} : \|\mathbf{x}\| = r\} \cup \{\mathbf{x} : \|\mathbf{x}\| = R\}$.

1 Curves

A *(space) curve* is a function $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$. The derivative of $\boldsymbol{\gamma}$ is also a vector, defined in the obvious way:

$$\boldsymbol{\gamma}'(t) = \lim_{h\to 0} \frac{\boldsymbol{\gamma}(t+h) - \boldsymbol{\gamma}(t)}{h}. \tag{1}$$

Natural parameter to work with is the *arc length*,

$$s(t) = \int_a^t \left\|\boldsymbol{\gamma}'(t)\right\| dt. \tag{2}$$

(Can check this is an invariant of the curve.) Write $\dot{f}$ for df/ds.

Tangent vector is

$$\mathbf{t} = \frac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}(t)\|} = \dot{\boldsymbol{\gamma}} \tag{3}$$

Normal vector is

$$\mathbf{n} = \frac{\dot{\mathbf{t}}}{\|\mathbf{t}\|}. \tag{4}$$

(Unsigned) *curvature* is then

$$\kappa = \left\|\dot{\mathbf{t}}\right\|, \tag{5}$$

so have

$$\dot{\mathbf{t}} = \kappa\mathbf{n}. \tag{6}$$

Binormal is simply

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}. \tag{7}$$

Define *torsion* by

$$\dot{\mathbf{b}} = -\kappa\mathbf{n}. \tag{8}$$

Torsion measures non-planarity.

If curve is smooth enough, $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are an ON basis of ^{3}: all defined to have unit length, $\mathbf{t}, \mathbf{n} \perp \mathbf{b}$ by definition, and have relations

$$\mathbf{t}\cdot\mathbf{t} = 1 \implies 0 = \mathbf{t}\cdot\dot{\mathbf{t}} = \kappa\mathbf{t}\cdot\mathbf{n} \tag{9}$$

$$\mathbf{b}\cdot\mathbf{b} = 1 \implies 0 = \mathbf{b}\cdot\dot{\mathbf{b}} \tag{10}$$

$$\mathbf{t}\cdot\mathbf{b} = 0 \implies 0 = \kappa\mathbf{n}\cdot\mathbf{b} + \mathbf{t}\cdot\dot{\mathbf{b}} = \mathbf{t}\cdot\dot{\mathbf{b}} \tag{11}$$

$$\mathbf{n}\cdot\mathbf{n} = 1 \implies 0 = \mathbf{n}\cdot\dot{\mathbf{n}} \tag{12}$$

$$\mathbf{n}\cdot\mathbf{t} = 0 \implies 0 = \dot{\mathbf{n}}\cdot\mathbf{t} + \mathbf{n}\cdot\dot{\mathbf{t}} = \dot{\mathbf{n}}\cdot\mathbf{t} + \kappa \tag{13}$$

$$\mathbf{n}\cdot\mathbf{b} = 0 \implies 0 = \dot{\mathbf{n}}\cdot\dot{\mathbf{b}} - \tau \tag{14}$$

First gives $\mathbf{t} \perp \mathbf{n}$, when $\kappa \neq 0$ so $\mathbf{n}$ is defined, and we have an ON basis. Second and third together imply $\dot{\mathbf{b}} \parallel \mathbf{n}$, so definition of τ makes sense. Last three imply $\dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b}$. Putting all these together gives the *Fresnet–Serret equations*, memorably written

$$\begin{pmatrix}\dot{\mathbf{t}}\\ \dot{\mathbf{n}}\\ \dot{\mathbf{b}}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}. \tag{15}$$

Can find expressions for κ and τ that do not rely on finding s as follows:

$$\boldsymbol{\gamma}' = \mathbf{t}s' \tag{16}$$

$$\boldsymbol{\gamma}'' = \kappa\mathbf{n}s'^2 + \mathbf{t}s'' \tag{17}$$

$$\boldsymbol{\gamma}''' = \dot{\kappa}\mathbf{n}s'^3 - \kappa^2\mathbf{t}s'^3 + \kappa\tau\mathbf{b}s'^3 + 3\kappa\mathbf{n}s's'' + \mathbf{t}s''' \tag{18}$$

$$= (s''' - \kappa^2 s'^3)\mathbf{t} + (\dot{\kappa}s'^3 + 3\kappa s's'')\mathbf{n} + \kappa\tau s'^3\mathbf{b}. \tag{19}$$

Then

$$\kappa = \frac{\|\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''\|}{s'^3} = \frac{\|\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''\|}{\|\boldsymbol{\gamma}'\|^3} \tag{20}$$

$$\tau = \frac{\|(\boldsymbol{\gamma}'\times\boldsymbol{\gamma}'')\cdot\boldsymbol{\gamma}'''\|}{\kappa^2 s'^6} = \frac{\|(\boldsymbol{\gamma}'\times\boldsymbol{\gamma}'')\cdot\boldsymbol{\gamma}'''\|}{\|\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''\|^2}. \tag{21}$$

The first of these also gives a definition of *signed curvature* in 2D:

$$k = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} \tag{22}$$

2 Integrals

We have two sorts of function that we can integrate in this course: scalar fields $\phi, \psi, \ldots : \mathbb{R}^3 \to \mathbb{R}$, and vector fields $\mathbf{F}, \mathbf{G}, \ldots : \mathbb{R}^3 \to \mathbb{R}^3$. If we can do the former, we can do the latter by creating scalar quantities using the scalar product.

2.1 Line Integral

Scalar The length of a curve L parametrised by $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^3$ is the integral of the arclength, $\int_L ds = \int_L \|\boldsymbol{\gamma}'(t)\|\, dt$. We can think of the scalar field as changing the weight of each point that we add up in this integral: this automatically gives us the right integration measure. Line integral is given by

$$\int_L \phi(\mathbf{x})\, ds = \int_a^b \phi(\boldsymbol{\gamma}(t))\left\|\boldsymbol{\gamma}'(t)\right\| dt. \tag{23}$$

(This is the only thing we could reasonably write, given how the integration should weight each line segment equally independent of parametrisation, and ϕ is a function of the point $\mathbf{x}$, which is parametrised by $\boldsymbol{\gamma}(t)$.)

Vector Sensible to start with thinking what we want this integral to do. Could compute an average value of the field over the line, which would be the same as doing (23) to each component. This would give something like the total force pushing L. But we want to compute things like the work done, which is about how much we have to push against (or are pushed along by) a force. This is determined by the scalar product of the vector field at $\boldsymbol{\gamma}(t)$ with the tangent vector of the curve at $\boldsymbol{\gamma}(t)$; this is then a scalar quantity that we integrate as usual:

$$\int_L \mathbf{F}(\mathbf{x}) \cdot \mathbf{t}\, ds = \int_a^b \mathbf{F}(\boldsymbol{\gamma}(t)) \cdot \frac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}'(t)\|} \left\|\boldsymbol{\gamma}'(t)\right\| dt. \tag{24}$$

We write $\boldsymbol{\gamma}'(t)\, dt = d\mathbf{L}$, and then the definition of integral is

$$\int_L \mathbf{F} \cdot d\mathbf{L} = \int_L \mathbf{F}(\mathbf{x}) \cdot \mathbf{t}\, ds = \int_a^b \mathbf{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt. \tag{25}$$

An alternative formulation is to separate this product into components: by the chain rule, $d\mathbf{L} = (dx, dy, dz)$, so if $\mathbf{F} = (P, Q, R)$, we can also write the integral as

$$\int_L \big(P(x,y,z)\, dx + Q(x,y,z)\, dy + R(x,y,z)\, dz\big). \tag{26}$$

2.2 Multiple Integrals

A *multiple integral* is, as it says on the tin, an integral over a multi-dimensional region. We start with the double integral; triple, quadruple and higher integrals can be defined in a similar way using n-dimensional cells instead of rectangles. The double integral is defined for continuous functions in the usual way, as the limit of a Riemann sum: we chop the region A up into shapes of size at most h, pick a point (x_i, y_i), and then the integral, if it exists, is

$$\iint_A f(x,y)\, dA = \lim_{h\to 0} \sum_{i=0}^{n(h)} f(x_i, y_i) A_i, \tag{27}$$

where A_i is the area of the ith rectangle. (Of course, it's normally easier for practical purposes to use equal rectangles in a grid, splitting this sum up into two sums, one in the x direction and the other in the y direction, but that involves dealing with the boundary explicitly, which we want to avoid in the first definition.

For practical purposes, if the area can be cut into slices of width dx that lie between curves $a(y)$ and $b(y)$, the integral can be evaluated as an iterated integral,

$$\iint_A f(x,y)\, dA = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)\, dx \right) dy. \tag{28}$$

There is nothing special about using x first: the same is true of y, so we could equally say

$$\iint_A f(x,y)\, dA = \int_a^b \left(\int_{c(x)}^{d(x)} f(x,y)\, dy \right) dx. \tag{29}$$

(The equality of these three integrals is known as *Fubini's Theorem*, and is not necessarily true if f is not continuous or the intervals are infinite.)

Changing Variables To change variables, we have to understand how the infinitesimal volume element changes. If the old coordinate system is (x, y, z) and the new one is (u, v, w), we find out how much an infinitesimal change in the old affects the new. We have

$$\begin{pmatrix} u(\mathbf{x}+d\mathbf{x}) \\ v(\mathbf{x}+d\mathbf{x}) \\ w(\mathbf{x}+d\mathbf{x}) \end{pmatrix} - \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \\ w(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} (\partial u/\partial x)\, dx + (\partial u/\partial y)\, dy + (\partial u/\partial z)\, dz \\ (\partial v/\partial x)\, dx + (\partial v/\partial y)\, dy + (\partial v/\partial z)\, dz \\ (\partial w/\partial x)\, dx + (\partial w/\partial y)\, dy + (\partial w/\partial z)\, dz \end{pmatrix} \tag{30}$$

$$= \begin{pmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{31}$$

$$= \left(\frac{\partial(u,v,w)}{\partial(x,y,z)} \right) \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{32}$$

This is a linear transformation of $d\mathbf{x}$, called the *Jacobian (matrix)*. As we know from Vectors and Matrices, the scale change in volume caused by a linear transformation is measured by the determinant, so the infinitesimal element changes by a factor of the *Jacobian (determinant)*,

$$du\, dv\, dw = \left| \frac{\partial(u,v,w)}{\partial(x,y,z)} \right| dx\, dy\, dz. \tag{33}$$

2.3 Surface Integral

We don't really know how to integrate over a surface, so we take a parametrisation of the surface, and integrate over the region where the parameters live (in $\mathbb{R}^2$) (we may have to use multiple parametrisations of different chunks if the surface is complicated). There are two things to deal with:

1. the region over which we integrate
2. the area element, i.e. how big the infinitesimal surface area is that corresponds to that in the parametrisation space.

In general, either the integrand or the definition of the parametrisation can be made nice, but the other one will be made nasty.

Scalar can again be inferred as weighting in the integral that calculates the area of the surface. What is this area? We work in the same way as changing variables in a multiple integral: suppose the parametrisation is $\boldsymbol{\sigma} : \mathbb{R}^2 \supset U \ni (u,v) \mapsto \boldsymbol{\sigma}(u,v) \in S \subset \mathbb{R}^3$. The area of the infinitesimal rectangle in the parameter set U is $du\, dv$. What does this become when we use $\boldsymbol{\sigma}$ to move to S? We can find this out by Taylor expansion:

$$\boldsymbol{\sigma}(u+du, v+dv) - \boldsymbol{\sigma}(u,v) = \frac{\partial \boldsymbol{\sigma}}{\partial u}(u,v)\, du + \frac{\partial \boldsymbol{\sigma}}{\partial v}(u,v)\, dv + o(du, dv).$$

Thus the infinitesimal vector (du, dv) undergoes a linear transformation, which can be written in matrix form as $(\boldsymbol{\sigma}_u \ \boldsymbol{\sigma}_v)$, where the subscripts are partial derivatives. The area is then scaled by the determinant of this transformation, which we recall from Vectors and Matrices is the same as $\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|$. The total surface area is therefore $\int_S dS = \int_U \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|\, du\, dv$, and finally we weight this at each point with the scalar field to arrive at

$$\int_S \phi(\mathbf{x})\, dS = \int_U \phi(\boldsymbol{\sigma}(u,v)) \, \|\boldsymbol{\sigma}_u(u,v) \times \boldsymbol{\sigma}_v(u,v)\|\, du\, dv \tag{34}$$

Vector Again we ask what we want this integral to do. The most useful idea is that of *flux*, i.e. how much stuff is passing through the surface. This is determined by how much of the flow is perpendicular to the surface, i.e. the scalar quantity $\mathbf{F} \cdot \mathbf{N}$, where $\mathbf{N}$ is a unit normal to S. There are two unit normals, so we define $\mathbf{N}$ as the one given by the direction of $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$; this is well-defined providing the surface is differentiable and *orientable*. Hence,

$$\int_S \mathbf{F}(\mathbf{x}) \cdot \mathbf{N}\, dS = \int_U \mathbf{F}(\boldsymbol{\sigma}(u,v)) \cdot \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} \, \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|\, du\, dv. \tag{35}$$

We write $\mathbf{N}\,dS = d\mathbf{S}$ for brevity, and the integral is then defined as

$$\int_S \mathbf{F}\cdot d\mathbf{S} = \int_S \mathbf{F}(\mathbf{x})\cdot\mathbf{N}\,dS = \int_U \mathbf{F}(\boldsymbol{\sigma}(u,v))\cdot\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v\,du\,dv. \quad (36)$$

2.4 Volume Integral

Volume integrals are theoretically simpler, mostly because they are just over 3D subsets of $\mathbb{R}^3$, so we need not change variables (and if we do, we just use the Jacobian as usual). Of course, we have to do a triple integral, which may be computationally nasty.

There are no nice vectors to associate to a volume either, so it is enough to confine ourselves to scalars. We have

$$\int_V \phi(\mathbf{x})\,dV = \int_V \phi(x,y,z)\,dx\,dy\,dz \quad (37)$$

N.B. It is common to write surface integrals with two integral signs, $\iint$, and volume integrals with three, $\iiint$. Integrals over closed curves and surfaces are also commonly written $\oint, \oiint$ respectively. We shall not do this, to avoid cluttering the notation, although it requires you to be more alert!

3 Vector Derivatives

3.1 Directional Derivative and Gradient

Directional Derivative In 1D, there is only one way to differentiate a function. In more dimensions, we have to look at the change in $\phi(\mathbf{x})$ as we go in a particular direction: this is the *directional derivative*, defined by

$$\nabla_{\mathbf{v}}\phi(\mathbf{x}) = \lim_{h\to 0}\frac{\phi(\mathbf{x}+h\mathbf{v})-\phi(\mathbf{x})}{h}. \quad (38)$$

This is also written as $\frac{\partial\phi}{\partial\mathbf{v}}$. It has the usual properties of 1D derivatives, being linear and satisfying Leibniz's rule. In particular, suppose we have an ON basis $\{\mathbf{e}_i\}$ at $\mathbf{x}$. Then

$$\nabla_{\mathbf{e}_i}\phi(\mathbf{x}) = \lim_{h\to 0}\frac{\phi(\mathbf{x}+h\mathbf{e}_1)-\phi(\mathbf{x})}{h} = \frac{\partial\phi}{\partial x_i}. \quad (39)$$

Differentiable A function $\phi:\mathbb{R}^n\to\mathbb{R}$ is *differentiable* if there is a linear map $L:\mathbb{R}^n\to\mathbb{R}$ that approximates it to $o(h)$, i.e.

$$\phi(\mathbf{x}+h\mathbf{v}) = \phi(\mathbf{x}) + hL(\mathbf{v}) + o(h). \quad (40)$$

L is then called the *derivative* of ϕ. Therefore if ϕ is differentiable, $L(\mathbf{v}) = \nabla_{\mathbf{v}}\phi(\mathbf{x})$.

Gradient A linear map $\mathbb{R}^n\to\mathbb{R}$ can always be represented by the dot product with a vector; for the derivative, this vector is called the *gradient*, $\nabla\phi(\mathbf{x})$ or $\operatorname{grad}\phi$,[1] so we also have $L(\mathbf{v}) = \mathbf{v}\cdot\nabla$. Therefore if ϕ is differentiable,

$$\nabla_{\mathbf{v}}\phi = L(\mathbf{v}) = \mathbf{v}\cdot\nabla\phi. \quad (41)$$

The gradient is a linear operator that maps scalar fields to vector fields. It also satisfies Leibniz's rule. It points in the direction in which ϕ increases fastest (as is clear: this is where $\mathbf{v}\cdot\nabla\phi$ is largest for a specific length of $\mathbf{v}$). Similarly, if $\mathbf{v}\cdot\nabla\phi$ is zero, then ϕ does not change to first order along the direction of $\mathbf{v}$; hence $\mathbf{v}$ is tangent to the level set of ϕ passing through x.

In coordinates To actually calculate the gradient, suppose we have an ON basis $\mathbf{e}_i$ at the point $\mathbf{x}$ as before. Then the components of $\nabla\phi$ are

$$(\nabla\phi)_i = \mathbf{e}_i\cdot\nabla\phi = \nabla_{\mathbf{e}_i}\phi = \frac{\partial\phi}{\partial x_i}. \quad (42)$$

This works in *any* set of orthogonal coordinates.

3.2 Divergence, Curl, Laplacian

We can generalise the directional derivative to vector fields, but there are other physically useful things that vector fields do (especially since increasing is not very meaningful in an unordered space like $\mathbb{R}^n$ for $n>1$). The two new derivatives we introduce here are ways of capturing this in $\mathbb{R}^3$.

Divergence We are interested in how much a vector field "spreads out" near $\mathbf{x}$. The sensible way to do this is to look how the flux through a small surface surrounding x differs from that at $\mathbf{x}$ itself, averaged by dividing by the volume inside the surface: i.e.

$$\operatorname{div}\mathbf{F}(\mathbf{x}) = \lim_{\operatorname{Vol}(V)\to 0}\frac{1}{\operatorname{Vol}(V)}\int_{\partial V}\mathbf{F}\cdot d\mathbf{S}, \quad (43)$$

where the normal points outwards.

Cartesian coordinates Suppose we take V to be a box of side lengths a,b,c parallel to $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$. Then, concentrating on the yz sides of the box,

$$\begin{aligned}
\operatorname{div}\mathbf{F}(\mathbf{x}) &= \lim_{abc\to 0}\frac{1}{abc}\left(\begin{array}{c}\displaystyle\int_{-b/2}^{b/2}\int_{-c/2}^{c/2}\big(F_1(\mathbf{x}+\tfrac{1}{2}a\mathbf{e}_1+y'\mathbf{e}_2+z'\mathbf{e}_3)\\ \qquad -F_1(\mathbf{x}-\tfrac{1}{2}a\mathbf{e}_1+y'\mathbf{e}_2+z'\mathbf{e}_3)\big)\,dy'\,dz'\\ +\cdots\end{array}\right)\\
&= \lim_{abc\to 0}\frac{1}{abc}\left(\int_{-b/2}^{b/2}\int_{-c/2}^{c/2}\left(a\frac{\partial F_1}{\partial x}(\mathbf{x}+y'\mathbf{e}_2+z'\mathbf{e}_3)+o(a)\right)dy'\,dz'+\cdots\right)\\
&= \lim_{abc\to 0}\left(\frac{1}{bc}\int_{-b/2}^{b/2}\int_{-c/2}^{c/2}\left(\frac{\partial F_1}{\partial x}(\mathbf{x}+y'\mathbf{e}_2+z'\mathbf{e}_3)\right)dy'\,dz'+\cdots\right)\\
&= \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)(\mathbf{x}),
\end{aligned}$$

or in other words,

$$\operatorname{div}\mathbf{F} = \frac{\partial F_i}{\partial x_i} = \nabla\cdot\mathbf{F}. \quad (44)$$

This expression is only valid in Cartesian coordinates.

Curl The *curl* of a vector field measures how much rotation there is locally. Since rotations are given by a vector along the axis and a magnitude (the angle), we expect the curl to be a vector. Given a unit vector $\mathbf{n}$, the component of $\operatorname{curl}\mathbf{F}$ in the direction of $\mathbf{n}$ is defined by the anticlockwise line integral around a small surface containing $\mathbf{x}$ with normal $\mathbf{n}$, divided by the area enclosed:

$$\mathbf{n}\cdot\operatorname{curl}\mathbf{F}(\mathbf{x}) = \lim_{\operatorname{Area}(S)\to 0}\frac{1}{\operatorname{Area}(S)}\int_{\partial S}\mathbf{F}\cdot d\mathbf{L}. \quad (45)$$

[1] The symbol ∇ is called "nabla", because it looks like a harp, but in this combination is read as "grad" or "gradient".

Cartesian Coordinates The simplest sort of curve to consider is a rectangle of sides a, b in one of the coordinate planes, aligned to the axes. We do the calculation of the $\mathbf{e}_3$-component, then generalise by cyclic permutation of the coordinate vectors. Most importantly, the line integral can be written so that opposite sides subtract:

$$\begin{aligned}
&\mathbf{e}_3 \cdot \operatorname{curl} \mathbf{F}(\mathbf{x}) \\
&= \lim_{ab\to 0} \frac{1}{ab} \left(\begin{array}{l} \displaystyle\int_{-b/2}^{b/2} (\mathbf{F}(\mathbf{x} + y'\mathbf{e}_2 + \tfrac{1}{2}a\mathbf{e}_1) - \mathbf{F}(\mathbf{x} + y'\mathbf{e}_2 - \tfrac{1}{2}a\mathbf{e}_1)) \cdot \mathbf{e}_2 \, dy' \\ \displaystyle - \int_{-a/2}^{a/2} (\mathbf{F}(\mathbf{x} + x'\mathbf{e}_1 + \tfrac{1}{2}b\mathbf{e}_2) - \mathbf{F}(\mathbf{x} + x'\mathbf{e}_1 - \tfrac{1}{2}b\mathbf{e}_2)) \cdot \mathbf{e}_1 \, dx' \end{array} \right) \\
&= \lim_{ab\to 0} \frac{1}{ab} \left(\int_{-b/2}^{b/2} a \frac{\partial F_2}{\partial x_1}(\mathbf{x} + y'\mathbf{e}_2)\, dy' - \int_{-a/2}^{a/2} b \frac{\partial F_1}{\partial x_2}(\mathbf{x} + x'\mathbf{e}_1)\, dx' \right) \\
&= \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right)(\mathbf{x}) \qquad (46)
\end{aligned}$$

Repeating for the other components, we find that

$$(\operatorname{curl} \mathbf{F})_i = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} = (\nabla \times \mathbf{F})_i, \qquad (47)$$

the latter of which is the notation we use from now on. Again, this expression is more complicated in other coordinate systems.

Laplacian The *Laplacian* is the most natural second-order differential operator that spits out the same type of field—vector or scalar—that it eats. In Cartesian coordinates, it has the simple expression

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \qquad (48)$$

The expression becomes more complicated in other coordinate systems. There are coordinate-free definitions, which differ between scalar and vector fields:

$$\Delta \phi = \nabla \cdot (\nabla \phi) \qquad (49)$$
$$\Delta \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}) \qquad (50)$$

3.3 Derivative Identities

The general rule for these is that if they're simple enough, you're probably better off guessing them than remembering, and if they're too complicated, they won't be used very much and are probably better derived than remembered.

All of these identities are coordinate-independent, so are true in any coordinate system.

Equivalents of the product rule:

$$\nabla(\phi\psi) = (\nabla\phi)\psi + \phi\nabla\psi \qquad (51)$$
$$\nabla \cdot (\phi\mathbf{F}) = (\nabla\phi) \cdot \mathbf{F} + \phi \nabla \cdot \mathbf{F} \qquad (52)$$
$$\nabla \times (\phi\mathbf{F}) = (\nabla\phi) \times \mathbf{F} + \phi(\nabla \times \mathbf{F}) \qquad (53)$$
$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \qquad (54)$$
$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} \qquad (55)$$
$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) \qquad (56)$$

The last two require the vector identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

Second derivative identities:

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \qquad (57)$$
$$\nabla \times (\nabla \phi) = 0 \qquad (58)$$
$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi = \Delta \phi \qquad (59)$$
$$\nabla(\nabla \cdot \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) + \Delta \mathbf{F} \qquad (60)$$

The easiest way to prove all of these is using indices. The last two are the definitions of the scalar and vector Laplacian.

3.4 Types of Vector Field

Vector field $\mathbf{F}$ is

Solenoidal if $\nabla \cdot \mathbf{F} = 0$.

Irrotational if $\nabla \times \mathbf{F} = 0$.

Conservative if $\int_L \mathbf{F} \cdot d\mathbf{L} = 0$ for any closed loop L. Stokes's Theorem implies that for simply connected regions, this is the same as irrotational (which is much easier to check, of course).

Poincaré lemmata For regions where any curve or sphere can be shrunk to a point (in two or three dimensions, such a space can have no holes in it, and is called *contractible* because it can be continuously shrunk to a point),

- If $\mathbf{F}$ is conservative, we can find a scalar field ϕ, the *scalar potential*, so that $\mathbf{F} = \nabla\phi$. (We can essentially do this by *defining* $\phi(b) - \phi(a) = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{L}$, since this is independent of path).
- If $\mathbf{F}$ is solenoidal, there is a vector field $\mathbf{A}$ so that $\mathbf{F} = \nabla \times \mathbf{A}$; this is called the *vector potential*. (This is rather harder to calculate.)

4 Orthogonal Curvilinear Coordinate Systems

Cartesian coordinates are often not the most sensible to use: the situation may have certain symmetries that make other coordinate systems more appropriate.

4.1 Cartesian Coordinates

Here, the basis vectors are $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, and point in the same directions at every point of space.

4.2 Cylindrical Coordinates

The basis vectors are

$$\hat{\mathbf{r}} = \cos\varphi\, \hat{\mathbf{x}} + \sin\varphi\, \hat{\mathbf{y}}, \qquad (61)$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}}, \qquad (62)$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}. \qquad (63)$$

The gradient, divergence and curl are given by

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{z}} \qquad (64)$$
$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z} \qquad (65)$$
$$\nabla \times \mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial\varphi} - \frac{\partial F_\varphi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\varphi) - \frac{\partial F_r}{\partial\varphi}\right)\hat{\mathbf{z}} \qquad (66)$$

The Laplacian can be found by combining the first two:

$$\Delta\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\phi) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}. \qquad (67)$$

One may also compute the vector Laplacian by similar means.

4.3 Spherical Coordinates

The basis vectors are

$$\hat{\mathbf{r}} = \cos\varphi\sin\theta\,\hat{\mathbf{x}} + \sin\varphi\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \tag{68}$$

$$\hat{\boldsymbol{\theta}} = \cos\varphi\cos\theta\,\hat{\mathbf{x}} + \sin\varphi\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \tag{69}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} \tag{70}$$

The gradient, divergence and curl are given by

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\boldsymbol{\phi}} \tag{71}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,F_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} \tag{72}$$

$$\begin{aligned}\nabla\times\mathbf{F} = {}& \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(F_\varphi\sin\theta\right) - \frac{\partial F_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} \\ & + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rF_\varphi\right)\right)\hat{\boldsymbol{\theta}} \\ & + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{\partial F_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}\end{aligned} \tag{73}$$

Similarly, the Laplacian is given by

$$\Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}. \tag{74}$$

4.4 *General Orthogonal Coordinates

In orthogonal curvilinear coordinates, the basis vectors vary from point to point. This variance must be taken into account in taking derivatives. Indeed, even the lengths of the basis vectors may change. We have to effectively start from scratch to define the quantities to do what we want. Let the coordinates be u_i.

Basis vectors and scale factors A good place to start is the expression for a general infinitesimal vector: we have, by the chain rule,

$$d\mathbf{x} = \sum_i \frac{\partial\mathbf{x}}{\partial u_i}du_i. \tag{75}$$

This should be expressible in terms of the basis at this point. What basis? We define the *basis vectors* $\mathbf{b}_i$ at $\mathbf{x}$ to be unit vectors in the direction of $\partial\mathbf{x}/\partial u_i$. (We will omit the $\mathbf{x}$ when the point is obvious.) The actual length of $\partial\mathbf{x}/\partial u_i$ is called the *scale factor* h_i. So, we have

$$\frac{\partial\mathbf{x}}{\partial u_i} = h_i\mathbf{e}_i, \tag{76}$$

and it is immediately apparent that we will have to discard summation convention.[2]

In the basis, a vector field may be written as $\mathbf{F} = \sum F_i\mathbf{e}_i$, where as usual $F_i = \mathbf{e}_i\cdot\mathbf{F}$.

In this new notation the infinitesimal increment becomes

$$d\mathbf{x} = \sum_i h_i\mathbf{e}_i du_i. \tag{77}$$

To do surface and volume integrals, we also need the surface elements and the volume element. The latter is easy, being constructed from the determinant of the independent incremented line elements,

$$dV = \det\left(\frac{\partial\mathbf{x}}{\partial u_1}du_1, \frac{\partial\mathbf{x}}{\partial u_2}du_2, \frac{\partial\mathbf{x}}{\partial u_3}du_3\right) = h_1h_2h_3du_1du_2du_3. \tag{78}$$

The surface area element for a surface perpendicular to $\mathbf{e}_k$ are just

$$\frac{\partial\mathbf{x}}{\partial u_i}du_i \times \frac{\partial\mathbf{x}}{\partial u_j}du_j = h_ih_jdu_idu_j\mathbf{e}_k. \tag{79}$$

Similarly, the arc length is

$$ds^2 = \sum_{i,j} h_ih_jdu_idu_j\mathbf{e}_i\cdot\mathbf{e}_j = \sum_i h_i^2du_i^2. \tag{80}$$

Gradient The gradient is now easy to find, since we know it is a scalar quantity relating the variation in ϕ to the variation in $\mathbf{x}$:

$$\phi(\mathbf{x}+d\mathbf{x}) - \phi(\mathbf{x}) = \nabla\phi(\mathbf{x})\cdot d\mathbf{x} + o(d\mathbf{x}). \tag{81}$$

But equally, Taylor's theorem gives

$$\phi(\mathbf{x}+d\mathbf{x}) - \phi(\mathbf{x}) = \sum_i \frac{\partial\phi}{\partial u_i}(\mathbf{x})du_i + o(du_i). \tag{82}$$

Then, expanding the first expression in the basis, we have

$$\nabla\phi\cdot d\mathbf{x} = \sum_i(\nabla\phi)_ih_idu_i, \tag{83}$$

where $(\nabla\phi)_i = \mathbf{e}_i\cdot\nabla\phi$. Since the increments du_i are independent, we equate coefficients between the last two expressions to find $\frac{\partial\phi}{\partial u_i} = (\nabla\phi)_ih_i$. Then,

$$\nabla\phi = \sum_i(\nabla\phi)_i\mathbf{e}_i = \sum_i\frac{1}{h_i}\frac{\partial\phi}{\partial u_i}\mathbf{e}_i. \tag{84}$$

Divergence We can either use the coordinate-free definition, or differentiate the vector field including the vectors in the basis (which requires identities like $\frac{\partial\mathbf{e}_i}{\partial u_j} = \frac{1}{h_i}\partial_ih_j$ for $i\neq j$, 0 for $i=j$), or write $\mathbf{F}$ in terms of $\nabla\mathbf{e}_i$, or use an identity from the next section,

$$\int_V\nabla\phi\cdot\mathbf{F}\,dV = \int_S\phi\mathbf{F}\cdot d\mathbf{S} - \int_V\phi\nabla\cdot\mathbf{F}\,dV. \tag{85}$$

We choose ϕ and S so that the surface integral is zero. We can also use ordinary integration by parts to find

$$\begin{aligned}\int_V\nabla\phi\cdot\mathbf{F}\,dV &= \int_V\sum_i\frac{1}{h_i}\frac{\partial\phi}{\partial u_i}F_i\left(\prod_j h_j\right)du_1\cdots du_n \\ &= \int_V\phi\sum_i\frac{\partial}{\partial u_i}\left(F_i\prod_{j\neq i}h_j\right)du_1\cdots du_n \\ &= \int_V\phi\frac{1}{\prod_k h_k}\sum_i\frac{\partial}{\partial u_i}\left(F_i\prod_{j\neq i}h_j\right)\left(\prod_k h_k\right)du_1\cdots du_n \\ &= \int_V\phi\frac{1}{\prod_k h_k}\sum_i\frac{\partial}{\partial u_i}\left(F_i\prod_{j\neq i}h_j\right)dV,\end{aligned}$$

from which it is clear that

$$\nabla\cdot\mathbf{F} = \frac{1}{\prod_k h_k}\sum_i\frac{\partial}{\partial u_i}\left(F_i\prod_{j\neq i}h_j\right). \tag{86}$$

Curl As usual, curl is annoying. One can do the same thing as with the divergence, but this time it is probably simpler to use the definition: considering the surface perpendicular to $\mathbf{e}_3$, we have line segments $h_1\,du_1\mathbf{e}_1$ and $h_2\,du_2\mathbf{e}_2$, and area element $h_1h_2\,du_1\,du_2$; so dotting with $\mathbf{F}$, dividing and taking the limit gives

$$(\nabla\times\mathbf{F})_3 = \frac{1}{h_1h_2}\left(\frac{\partial}{\partial u_1}(h_2F_2) - \frac{\partial}{\partial u_2}(h_1F_1)\right), \tag{87}$$

and similarly for the other components.

[2] This is partially a consequence of sticking to a normalised basis, and partially because the basis is assumed to be orthogonal: in general the inner products $(\partial_i\mathbf{x})\cdot(\partial_j\mathbf{x})$ form a symmetric matrix $g_{ij}(\mathbf{x})$ called the *metric tensor*.

Scalar Laplacian Of course, the Laplacian is obtained from combining the gradient and divergence formulae:

$$\Delta \phi = \frac{1}{\prod_k h_k} \sum_i \frac{\partial}{\partial u_i} \left(\frac{\prod_{j \neq i} h_j}{h_i} \frac{\partial \phi}{\partial u_i} \right). \tag{88}$$

5 Fundamental Theorems of Vector Calculus

For line integrals, the ordinary Fundamental Theorem of Calculus gives

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla \phi \cdot d\mathbf{x} = \phi(\mathbf{b}) - \phi(\mathbf{a}). \tag{89}$$

for any piecewise-differentiable path and once-differentiable ϕ.

5.1 Divergence Theorem

Theorem (Divergence Theorem). *Let S be a piecewise-differentiable orientable* closed *surface enclosing a volume V, $\mathbf{F}$ a differentiable vector field. Then*

$$\int_V \nabla \cdot \mathbf{F} \, dV = \int_S \mathbf{F} \cdot \, d\mathbf{S}, \tag{90}$$

where $d\mathbf{S} = \mathbf{N} \, dS$ and $\mathbf{N}$ points outwards, away from V.

(AKA Gauss's Theorem)

Sketch proof. The right-hand side looks like the definition of the divergence, but over a large volume. So what we do is split the volume up into n small cells V_i of volume δV with boundaries S_i. Then the right-hand side becomes

$$\int_S \mathbf{F} \cdot \, d\mathbf{S} = \sum_{i=1}^{n} \int_{S_i} \mathbf{F} \cdot \, d\mathbf{S}_i,$$

because when two cells are adjacent, the surface is the same, the function is the same, but the normals point in opposite directions, so the integrals cancel with each other.

Now, when δV is small enough, the definition of divergence allows us to write

$$\int_{S_i} \mathbf{F} \cdot \, d\mathbf{S}_i = (\nabla \cdot \mathbf{F}(\mathbf{x}) \delta V + o(\delta V),$$

where $\mathbf{x}_i$ is a point in V_i. Summing,

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \sum_{i=1}^{n} (\nabla \cdot \mathbf{F}(\mathbf{x}_i)) \delta V + o(\delta V) = \left(\sum_{i=1}^{n} (\nabla \cdot \mathbf{F}(\mathbf{x}_i)) \delta V \right) + o(n \delta V).$$

As $\delta V \to 0$ and $n \to \infty$, the bracket tends to $\int_V \nabla \cdot \mathbf{F} \, dV$, and we just have to show the error term tends to zero. This is the case because we must have $n \delta V = V$ constant, so in fact the error is $o(1)$. □

Notice that the direction of the normal is an immediate consequence of the direction in the definition of the divergence.

5.2 Stokes's Theorem

Theorem (Stokes's Theorem). *Let L be a piecewise-differentiable simple closed curve bounding an orientable piecewise-differentiable surface S, $\mathbf{F}$ a differentiable vector field. Then*

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_L \mathbf{F} \cdot d\mathbf{L}. \tag{91}$$

Sketch proof. The principle is identical to the proof of the Divergence Theorem: divide S up into n small surfaces S_i of area δS with boundaries L_i. Then

$$\int_L \mathbf{F} \cdot d\mathbf{L} = \sum_{i=1}^{n} \int_{L_i} \mathbf{F} \cdot d\mathbf{L}_i,$$

because when two cells meet in an edge, the integrand has the same value but the edge is traversed in the opposite directions, so the integrals cancel.

For small δS, the definition of curl then allows us to write

$$\int_{L_i} \mathbf{F} \cdot d\mathbf{L}_i = \mathbf{N}_i \cdot (\nabla \times \mathbf{F}(\mathbf{x}_i)) \delta S + o(\delta S),$$

where $\mathbf{x}_i$ is a point in S_i and $\mathbf{N}_i$ is the normal to S at $\mathbf{x}_i$ chosen to be compatible with the one in the definition of curl. The total sum is then

$$\sum_{i=1}^{n} \int_{L_i} \mathbf{F} \cdot d\mathbf{L}_i = \left(\sum_{i=1}^{n} \mathbf{N}_i \cdot (\nabla \times \mathbf{F}(\mathbf{x}_i)) \delta S \right) + o(n \delta S),$$

and once again as $\delta S \to 0$, $n \to \infty$, the first term tends to $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, and the order term is $o(1)$ because $n \delta S = S$. □

5.3 Green's Theorem in the Plane

Let S be a connected region of $\mathbb{R}^2$ bounded by a piecewise-differentiable simple closed curve L. Setting $\mathbf{F} = (M, N, 0)$ and $d\mathbf{S} = \mathbf{e}_3 \, dx \times dy$ in Stokes's Theorem (91), we have $d\mathbf{L} = (dx, dy)$ and so

$$\int_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy = \int_L (M \, dx + N \, dy). \tag{92}$$

(This also follows from the Divergence Theorem in two dimensions, the outward normal being $(dy, -dx)$, the vector field $(M, -L)$.)

A nice application is that $M = y/2$, $N = x/2$ makes the left-hand side $\int_S 1 \, dx \, dy$, so

$$\text{Area inside } L = \frac{1}{2} \int_L (y \, dx - x \, dy). \tag{93}$$

5.4 Green's Identities

Various product rules analogous to those in § 3.3 may be used to produce analogues of integration by parts for vector derivatives: we have

$$\nabla \cdot (\phi \nabla \psi) = \phi \, \Delta \, \psi + (\nabla \phi) \cdot (\nabla \psi), \tag{94}$$

which gives *Green's first identity*,

$$\int_V (\phi \, \Delta \, \psi + (\nabla \phi) \cdot (\nabla \psi)) \, dV = \int_S \phi (\nabla \psi) \cdot d\mathbf{S}. \tag{95}$$

We can interchange ϕ and ψ in (94) and subtract to find

$$\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \, \Delta \, \psi - \psi \, \Delta \, \phi, \tag{96}$$

and integrating gives *Green's Second Identity*,

$$\int_V (\phi \, \Delta \, \psi - \psi \, \Delta \, \phi) \, dV = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\mathbf{S}. \tag{97}$$

5.5 Example: Fluid Dynamics

Let ρ be the density, $\mathbf{u}$ the velocity of a fluid. The Euler equations for frictionless fluid flow are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \qquad \text{(Mass conservation)} \tag{98}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \rho \mathbf{g} \quad \text{(Momentum conservation)}, \tag{99}$$

where p is the pressure and $\mathbf{g}$ the acceleration due to gravity.

Integrated Mass Conservation Applying the Divergence Theorem to the first equation gives

$$\frac{d}{dt}\int_V \rho\, dV = -\int_S \rho\mathbf{u}\cdot d\mathbf{S}, \tag{100}$$

so the mass in a volume decreases due to mass flowing out of the surface.

Vorticity We define the *vorticity* $\boldsymbol{\omega} = \nabla\times\mathbf{u}$. Suppose ρ is constant. Then $\nabla\cdot\mathbf{u} = 0$. We have, using vector identities,

$$\nabla\cdot\boldsymbol{\omega} = 0 \tag{101}$$

$$\mathbf{u}\times\boldsymbol{\omega} = \nabla(\tfrac{1}{2}u^2) - (\mathbf{u}\cdot\nabla)\mathbf{u}, \tag{102}$$

and then the momentum equation can be written as

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \nabla(\tfrac{1}{2}u^2) - \mathbf{u}\times\boldsymbol{\omega}\right) = -\nabla p + \rho\mathbf{g}. \tag{103}$$

We also have

$$\nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = \mathbf{u}(\nabla\cdot\boldsymbol{\omega}) + (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - \boldsymbol{\omega}(\nabla\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$$

by the earlier equations. Assuming $\mathbf{g}$ is constant, taking the curl of the momentum equation gives

$$\frac{\partial\boldsymbol{\omega}}{\partial t} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = 0, \tag{104}$$

the *vorticity equation.*

Bernoulli equation Continue to assume that ρ is constant. Assume that $\boldsymbol{\omega} = 0$ initially. Then $\partial\boldsymbol{\omega}/\partial t = 0$ from the vorticity equation, and so $\boldsymbol{\omega}$ remains zero. Hence $\nabla\times\mathbf{u} = 0$, so $\mathbf{u}$ can be written as $\nabla\phi$ for some potential ϕ. Then the momentum equation becomes

$$\rho\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2\right) = -\nabla p + \nabla(\rho\mathbf{g}\cdot\mathbf{x}).$$

Everything in sight is a gradient, so integrating this up gives

$$\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2\right) = -p + \rho\mathbf{g}\cdot\mathbf{x} + f(t), \tag{105}$$

where $f(t)$ is an integration constant; this is called the *Bernoulli equation.*

5.6 Example: Maxwell's Equations

Maxwell's equations for electromagnetism are

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss's Law)} \tag{106}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \text{(No magnetic monopoles)} \tag{107}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \text{(Faraday's Law of Induction)} \tag{108}$$

$$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \qquad \text{(Ampère–Maxwell Law)} \tag{109}$$

$\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, ρ the charge density, $\mathbf{J}$ the current density, and μ_0 and ϵ_0 constants called the *permeability of free space* and *permitivity of free space.*

Integral forms Using the Divergence Theorem, we can write down integral forms of the first two:

$$\int_{\partial V}\mathbf{E}\cdot d\mathbf{S} = \frac{1}{\epsilon_0}\int_V \rho\, dV \tag{110}$$

$$\int_{\partial V}\mathbf{B}\cdot d\mathbf{S} = 0. \tag{111}$$

These say that the electric flux through a closed surface is determined by the charge inside it, whereas, since there are no magnetic charges, the total magnetic flux through a closed surface is zero.

Stokes's Theorem gives us integral forms of the last two:

$$\int_{\partial S}\mathbf{E}\cdot d\mathbf{L} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{112}$$

$$\int_{\partial S}\mathbf{B}\cdot d\mathbf{L} = \mu_0\int_S \mathbf{J}\cdot d\mathbf{S} + \mu_0\epsilon_0\frac{d}{dt}\int_S \mathbf{E}\cdot d\mathbf{S}. \tag{113}$$

Continuity equation Differentiating Gauss's Law wrt t and using the Ampère–Maxwell Law gives

$$\frac{\partial\rho}{\partial t} = \epsilon_0\frac{\partial}{\partial t}\nabla\cdot\mathbf{E} = \nabla\cdot\left(\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) = \nabla\cdot\left(\frac{1}{\mu_0}\nabla\times\mathbf{B} - \mathbf{J}\right) = -\nabla\cdot\mathbf{J}, \tag{114}$$

the *continuity equation* for charge and current density. Integrating and using the Divergence Theorem gives

$$\frac{d}{dt}\int_V \rho\, dV = -\int_{\partial V}\mathbf{J}\cdot d\mathbf{S}, \tag{115}$$

i.e. that the rate of change of charge inside a volume is the amount of current flowing in.

Conservation of Energy We assert that the energy density is given by $\mathcal{E} = \frac{\epsilon_0}{2}\|\mathbf{E}\|^2 + \frac{1}{2\mu_0}\|\mathbf{B}\|^2$. Then

$$\begin{aligned}\frac{\partial\mathcal{E}}{\partial t} &= \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu_0}\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{B}\\ &= \mathbf{E}\cdot\left(-\mathbf{J} + \frac{1}{\mu_0}\nabla\times\mathbf{B}\right) - \frac{1}{\mu_0}(\nabla\times\mathbf{E})\cdot\mathbf{B}\\ &= -\mathbf{E}\cdot\mathbf{J} - \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})\end{aligned}$$

by using an identity for the divergence. Integrating this over a fixed volume and using the divergence theorem,

$$\frac{d}{dt}\int_V \mathcal{E}\, dV = -\int_V \mathbf{E}\cdot\mathbf{J}\, dV - \frac{1}{\mu_0}\int_S(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S}, \tag{116}$$

which says that energy is lost by ohmic heating (first term on RHS) and carried away by the vector $(\mathbf{E}\times\mathbf{B})/\mu_0$, which is called the *Poynting vector.*

Electromagnetic waves Suppose we are *in vacuo*, with no charge or current. Taking the curl of the last two equations and using each other, we have

$$-\nabla\times(\nabla\times\mathbf{B}) = -\epsilon_0\mu_0\frac{\partial}{\partial t}\nabla\times\mathbf{E} = \epsilon_0\mu_0\frac{\partial^2\mathbf{B}}{\partial t^2}, \tag{117}$$

$$-\nabla\times(\nabla\times\mathbf{E}) = \frac{\partial}{\partial t}\nabla\times\mathbf{B} = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{118}$$

The first two equations allow us to rewrite the left-hand sides as vector Laplacians, and we have

$$\left(\Delta - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = 0, \quad \left(\Delta - \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = 0, \tag{119}$$

so $\mathbf{E}$ and $\mathbf{E}$ satisfy wave equations with speed $c = (\epsilon_0\mu_0)^{-1/2}$. One finds this quantity experimentally to be very close to the speed of light, leading Maxwell to theorise that light is electromagnetic radiation.

Potentials The lack of magnetic monopoles implies that $\mathbf{B}$ is solenoidal. Hence we can find a vector potential $\mathbf{A}$ so $\mathbf{B} = \nabla\times\mathbf{A}$. $\mathbf{E}$ is more complicated, but with this new form for $\mathbf{B}$, we can rewrite Faraday's Law as

$$\nabla\times\left(\mathbf{E} + \frac{\partial A}{\partial t}\right) = 0, \tag{120}$$

and so $\mathbf{E} + \frac{\partial A}{\partial t}$ is conservative. Therefore it can be represented as the gradient of a scalar potential $-\phi$. We obtain the potential representations

$$\mathbf{E} = -\nabla\phi - \frac{\partial A}{\partial t}, \quad \mathbf{B} = \nabla\times\mathbf{A}, \tag{121}$$

which automatically satisfy the middle two equations. The remaining ones can be written in terms of the potentials as

$$-\Delta\phi - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = \frac{\rho}{\epsilon_0} \tag{122}$$

$$\left(-\Delta + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = \mu_0\mathbf{J}. \tag{123}$$

6 Laplace's Equation

Laplace's equation is

$$\Delta\phi(x) = 0 \quad x \in D. \tag{124}$$

This is satisfied in a region $D \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$, and we impose boundary conditions on ∂D. Common ones are

$$\phi(x) = f(x) \qquad x \in \partial D \quad \text{(Dirichlet)}, \tag{125}$$

$$\frac{\partial\phi}{\partial n}(x) = g(x) \qquad x \in \partial D \quad \text{(Neumann)}, \tag{126}$$

$$\alpha(x)\phi(x) + \frac{\partial\phi}{\partial n}(x) = h(x) \qquad x \in \partial D \quad \text{(Robin)}, \tag{127}$$

($\alpha > 0$), in increasing order of difficulty.

6.1 Uniqueness

Suppose that ϕ solves Laplace's equation on D with Dirichlet BCs (125). Then ϕ is unique.

Proof. Suppose not. Then there is $\psi \neq \phi$ satisfying (124) and (125). Write $w = \phi - \psi$. Then $w \neq 0$, and w satisfies

$$\Delta w(x) = 0 \qquad x \in D$$
$$w(x) = 0 \qquad x \in \partial D.$$

By Green's first identity (95), we have

$$\int_D |\nabla w|^2\, dV = \int_{\partial D} w\frac{\partial w}{\partial n}\, dS - \int_D w\,\Delta w\, dV = 0,$$

from the conditions on w. Hence w is constant, and the boundary conditions imply it is zero. # □

It is clear that the same technique gives that *the solution to Laplace's equation with Neumann BCs is unique up to an additive constant.*

6.2 Maximum principle

Mean value formula Let ϕ solve Laplace's equation. Apply the Divergence Theorem to a ball of radius R centred at $\mathbf{a}$, using spherical coordinates centred at a:

$$0 = \int_{\|\mathbf{x}-\mathbf{a}\|<R} \nabla\cdot(\nabla\phi)\, dV = \int_{\|\mathbf{x}-\mathbf{a}\|=R} \hat{\mathbf{r}}\cdot\nabla\phi\, dS$$
$$= \int_{\|\mathbf{x}-\mathbf{a}\|=R} \frac{\partial\phi}{\partial r}\, dS = \frac{d}{dr}\int_{\|\mathbf{x}-\mathbf{a}\|=R} \phi\, dS,$$

and so $\int_{\|\mathbf{x}\|=R}\phi\, dS$ is independent of R. Dividing by $4\pi R^2$ and taking $R \to 0$, we find the *mean value formula*

$$\phi(\mathbf{a}) = \frac{1}{4\pi R^2}\int_{\|\mathbf{x}-\mathbf{a}\|=R} \phi(\mathbf{x})\, dS, \tag{128}$$

i.e. that the value of a harmonic function at the centre of a ball is the average of its values on its surface. Multiplying by $4\pi R^2$ and integrating from 0 to R gives another formula, as the average over the volume:

$$\phi(\mathbf{a}) = \frac{3}{4\pi R^3}\int_{\|\mathbf{x}-\mathbf{a}\|<R} \phi(\mathbf{x})\, dV, \tag{129}$$

Suppose that ϕ solves Laplace's equation in an open region V. If ϕ has a local maximum on the interior of V, then ϕ is constant.

Proof. Suppose that ϕ is not constant and $\mathbf{a}$ is an interior local maximum of ϕ. Then by definition there is a small sphere $S(a,\varepsilon)$ of radius ε on which $\phi(\mathbf{a}) > \phi(\mathbf{x})$. But then

$$\phi(\mathbf{a}) = \frac{1}{4\pi\varepsilon^2}\int_{S(a,\varepsilon)} \phi(\mathbf{x})\, dS \leqslant \sup_{\mathbf{x}\in S(a,\varepsilon)} \phi(\mathbf{x}) < \phi(\mathbf{a}), \tag{130}$$

which is a contradiction. Hence if ϕ has an interior local maximum, ϕ is constant. □

6.3 Poisson's Equation

Poisson's equation is "Laplace with something on the other side",

$$-\Delta\phi = f. \tag{131}$$

(The sign convention is different in different subjects.) Since the difference of two solutions satisfies Laplace's equation, the reasoning of the last section also applies here, and we find that solutions to Poisson's equation with one of the boundary conditions are unique (or unique up to a constant in the Neumann case).

Gauss's Method Uniqueness is all very well, but how do we solve it? Suppose first $f = f(r)$ is spherically symmetric. Then integrating both sides of the equation over the ball of radius R and applying the Divergence Theorem, we have

$$\int_{\|\mathbf{x}\|<R} f\, dV = -\int_{\|\mathbf{x}\|<R} \nabla\cdot(\nabla\phi)\, dV$$
$$= -\int_{\|\mathbf{x}\|=R} \hat{\mathbf{r}}\cdot\nabla\phi\, dS$$
$$= -\int_{\|\mathbf{x}\|=R} \frac{\partial\phi}{\partial r}\, dS$$

This holds for any ϕ and f. If we now assume that f is spherically symmetric, the left-hand side becomes

$$\int_{r=0}^{R}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} f(r)r^2\sin\theta\, d\theta\, d\phi\, dr = 4\pi\int_0^R r^2 f(r)\, dr.$$

If we assume that ϕ is spherically symmetric, the integral on the right becomes $4\pi R^2\frac{d\phi}{dr}(R)$, and we find

$$\frac{d\phi}{dr}(R) = -\frac{1}{R^2}\int_0^R r^2 f(r)\, dr, \tag{132}$$

so

$$\phi(\rho) = \phi(0) - \int_0^\rho \frac{1}{R^2}\left(\int_0^R r^2 f(r)\, dr\right) dR. \tag{133}$$

Spherical Shell Suppose all the mass is concentrated in a shell of radius R and total mass M. Then inside, the Divergence Theorem implies that $-4\pi r^2\phi'(r)$ is zero, so ϕ is constant. Outside, $-4\pi r^2\phi'(r) = M$, so we find

$$\phi(r) = \frac{M}{4\pi r} + A, \tag{134}$$

A a constant that we can safely set to zero. The potential inside is found by continuity to be $M/(4\pi R)$.

In particular, the potential outside the sphere does not depend on R; this is called *Newton's theorem*. Taking $R \to 0$, we obtain the potential of a point particle of mass M,

$$\phi(r) = \frac{M}{4\pi r}. \tag{135}$$

Solid Sphere One finds the potential of a solid uniform-density sphere either by going through the same procedure, or by adding up the potentials corresponding to shells of radius $0 < r < R$; in both cases it is easier to work with the mass density ρ, and then relate it to the total mass by $M = \rho V$.

Cylindrical Symmetry In the case of an f with cylindrical symmetry, one can take a cylinder of height z and radius R and apply the divergence theorem. Assuming ϕ only depends on r, the disks on either end contribute nothing, and we obtain

$$-2\pi Rz\phi'(R) = 2\pi z \int_0^R rf(r)\,dr, \tag{136}$$

so

$$\phi'(R) = -\frac{1}{R}\int_0^R rf(r)\,dr, \tag{137}$$

In the same way as before, one finds that an infinitely long line with mass density λ has potential $-\frac{\lambda}{2\pi}\log r$.

Integral Solution Since Poisson's equation is linear, one may be led to think that one could think of a general function $f(\mathbf{x})$ as an infinite sum, or rather, and integral, of point charges at points $\mathbf{x}'$, each of weight $f(\mathbf{x}')$, i.e.

$$f(\mathbf{x}) = \int_{\mathbb{R}^3} f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}', \tag{138}$$

Then, the solution to Poisson's equation is given adding up the solutions for each of these point charges, i.e.

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{f(\mathbf{x}')}{4\pi\,\|\mathbf{x}-\mathbf{x}'\|}\,d\mathbf{x}'. \tag{139}$$

This does work, provided f decays fast enough at ∞ for the integral to converge; this will be discussed properly in IB Methods.

6.4 *Helmholtz Decomposition

Any vector field in a finite region, or any sufficiently-fast-decaying vector field in an infinite region, can be written as the sum of a solenoidal and an irrotational part,

$$\mathbf{F} = -\nabla\phi + \nabla\times\mathbf{A}. \tag{140}$$

This looks a bit bonkers, but requires only being able to invert the vector Laplacian; that is, to find a solution to the equation

$$-\Delta\,\mathbf{G} = \mathbf{F}. \tag{141}$$

Why does this help? Well,

$$-\Delta\,\mathbf{G} = -\nabla(\nabla\cdot\mathbf{G}) + \nabla\times(\nabla\times\mathbf{G}), \tag{142}$$

and of course the first term is irrotational and the second is solenoidal as we want.

How do we find $\mathbf{G}$? Well, it suffices to take each component and solve the scalar Poisson equation.

7 Cartesian Tensors in $\mathbb{R}^3$

Given two right-handed, orthonormal systems of coordinates $\mathbf{e}_i, \mathbf{e}'_{i'}$, there is a matrix $R_{i'j}$ so that $\mathbf{e}'_{i'} = R_{i'j}\mathbf{e}_j$. (For clarity, we put primes on the indices in a primed basis.) Dotting with e_k, we find that $R_{i'k} = \mathbf{e}'_{i'}\cdot\mathbf{e}_k$. The reverse of this argument and the symmetry of the dot product imply that $\mathbf{e}_i = R_{ij'}\mathbf{e}'_{j'}$. Combining these, we find that $e_i = R_{ij'}\mathbf{e}'_{j'} = R_{ij'}R_{j'k}\mathbf{e}_k$, so

$$R_{ij'}R_{j'k} = \delta_{ik} \tag{143}$$

Hence $R_{i'j}$ is an orthogonal matrix.

The vector $\mathbf{v} = v_i\mathbf{e}_i = v'_{j'}\mathbf{e}'_{j'}$ is the same object in both bases. Inserting the transformation law, $v_i\mathbf{e}_i = v'_{j'}R_{j'k}\mathbf{e}_k$, and dotting with $\mathbf{e}_l$, the components transform as

$$v_i = R_{j'i}v'_{j'} \implies v'_{j'} = R_{ij'}v_i = (R^{-1})_{j'i}v_i, \tag{144}$$

the opposite to the basis vectors.

A *tensor* of rank p is a linear map $T : V^p \to \mathbb{R}$. In components, it looks like an object $T_{i_1\cdots i_p} = T(\mathbf{e}_{i_1},\dots,\mathbf{e}_{i_p})$ with p indices that transforms as

$$T'_{i'_1\cdots i'_p} = R_{i'_1j_1}\cdots R_{i'_pj_p}T_{j_1\cdots j_p}. \tag{145}$$

The entire point of this definition is that *scalars are invariant under change of basis.* Hence we find out the transformation law by saying that

$$\begin{aligned} T(\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_p) &= T(\mathbf{e}_{i_1},\dots,\mathbf{e}_{i_p})v_{i_1}\cdots v_{i_p} && (146)\\ &= T_{i_1\cdots i_p}v_{i_1}\cdots v_{i_p} && (147)\\ &= T'_{i'_1\cdots i'_p}v'_{i'_1}\cdots v'_{i'_p} && (148)\end{aligned}$$

is the same in both bases. Now, we know how the components v_i transform, so we can write down the new components in terms of the old:

$$T'_{j'_1\cdots j'_p}v'_{j'_1}\cdots v'_{j'_p} = T_{i_1\cdots i_p}v_{i_1}\cdots v_{i_p} = T_{i_1\cdots i_p}R_{j'_1i_1}v'_{j'_1}\cdots R_{j'_pi_p}v'_{j'_p}. \tag{149}$$

Subtracting, the only linear map that gives zero for every input is exactly zero, so we must have

$$T'_{j'_1\cdots j'_p} = R_{j'_1i_1}\cdots R_{j'_ki_p}T_{i_1\cdots i_p} \tag{150}$$

Alternatively, we can compute using the definition of the components:

$$\begin{aligned} T'_{j'_1\cdots j'_p} &= T(\mathbf{e}'_{j'_1},\dots,\mathbf{e}'_{j'_p}) && (151)\\ &= T(R_{j'_1i_1}\mathbf{e}_{i_1},\dots,R_{j'_pi_p}\mathbf{e}_{i_p}) && (152)\\ &= R_{j'_1i_1}\cdots R_{j'_pi_p}T(\mathbf{e}_{i_1},\dots,\mathbf{e}_{i_p}) && (153)\\ &= R_{j'_1i_1}\cdots R_{j'_pi_p}T_{i_1\cdots i_p}. && (154)\end{aligned}$$

by linearity.

N.B. Both second-rank tensors and rotation matrices have two indices and can be written as a two-dimensional array of numbers, but they are not the same thing: rotation matrices have "one foot in each basis", since one of their indices is primed and the other is unprimed, whereas a tensor's components live in one specific basis.

7.1 Tensor Operations

Since everything is linear, the sum of two p-tensors is a p-tensor,

$$(S+T)(\mathbf{v}_1,\dots,\mathbf{v}_p) = S(\mathbf{v}_1,\dots,\mathbf{v}_p) + T(\mathbf{v}_1,\dots,\mathbf{v}_p), \tag{155}$$

or in components,

$$(S+T)_{i_1\cdots i_p} = S_{i_1\cdots i_p} + T_{i_1\cdots i_p} \tag{156}$$

Tensors hav two products: given a p-tensor $S_{i_1\cdots i_p}$ and a q-tensor $T_{j_1\cdots j_q}$, their *outer product* $S_{i_1\cdots i_p}T_{j_1\cdots j_q}$ is a $(p+q)$-tensor.

A new operation for tensors is an inner product called *contraction*: given a p-tensor, we can feed two of the slots an element of an orthonormal basis and sum over the basis to get a $(p-2)$-tensor. This is easily expressed in components as

$$T_{i_1\cdots i_kji_{k+1}\cdots iji_{l+1}\cdots i_{p-2}} \tag{157}$$

The *trace* of a 2-tensor is a scalar, that agrees with the definition we know about for matrices, i.e. T_{ii}. Being a scalar, it is basis-independent.

7.2 Quotient Theorem

Suppose that $A_{i_1\cdots i_p}$ is an arbitrary tensor and $B_{j_1\cdots j_r} = C_{j_1\cdots j_r i_1\cdots i_p} A_{i_1\cdots i_p}$ is always a tensor. Then $C_{j_1\cdots j_r i_1\cdots i_p}$ is a tensor.

The proof works in exactly the same way as all other proofs of this type: transforming, we have

$$\begin{aligned} &C'_{j'_1\cdots j'_r i'_1\cdots i'_p} A'_{i'_1\cdots i'_p} = B'_{j'_1\cdots j'_r} \\ &= R_{j'_1 k_1}\cdots R_{j'_r k_r} B_{k_1\cdots k_r} \\ &= R_{j'_1 k_1}\cdots R_{j'_r k_r} C_{k_1\cdots k_r m_1\cdots m_p} A_{m_1\cdots m_p} \\ &= R_{j'_1 k_1}\cdots R_{j'_r k_r} C_{k_1\cdots k_r m_1\cdots m_p} R_{i'_1 m_1}\cdots R_{i'_p m_p} A'_{i'_1\cdots i'_p}, \end{aligned}$$

the last line using the inverse transformation on $A_{m_1\cdots m_p}$. This is true for arbitrary components, so we therefore have

$$C'_{j'_1\cdots j'_r i'_1\cdots i'_p} = R_{j'_1 k_1}\cdots R_{j'_r k_r} R_{i'_1 m_1}\cdots R_{i'_p m_p} C_{k_1\cdots k_r m_1\cdots m_p}, \tag{158}$$

as required.

7.3 Isotropic Tensors

An *isotropic tensor* is the same in any coordinate system. In this course we only discuss the rank 1, -2 and -3 ones in three dimensions.

Rank 0 Obviously every scalar is isotropic: that's the point.

Rank 1 There are no nonzero isotropic vectors, since for any given vector, we can always find a rotation by π that changes it.

Rank 2 Suppose a_{ij} is an isotropic tensor. Then $a_{i'j'} = R_{i'i} R_{j'j} a_{ij}$. It is in fact enough to look at a rotation by $\pi/2$, e.g.

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This gives $a_{13} = a_{31} = a_{23} = a_{32} = 0$, $a_{11} = a_{22}$, and $a_{12} + a_{21} = 0$. But the indices are symmetrical, so we can do the same with a rotation about the x-axis, which gives $a_{12} = a_{21} = 0$ and $a_{22} = a_{33}$. Hence the only possible isotropic 2-tensor is $\lambda\delta_{ij}$. It is easy to check this really is isotropic in the usual way.

Higher rank Extension of this process gives that isotropic 3-tensors are proportional to ε_{ijk}, and of rank 4, there are three, so the general case is

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}. \tag{159}$$

The product of two isotropic tensors is isotropic. This, of course, implies that $\epsilon_{mij}\epsilon_{mkl}$ is made out of the rank-four tensors. If $i = j \neq k = l$, then $0 = \lambda$. Putting everything equal gives $\mu = \nu$. Choosing a nonzero component to find the scale then gives the familiar

$$\epsilon_{mij}\epsilon_{mkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{160}$$

Isotropic tensors are useful for evaluating integrals. For example,

$$a_{ij} = \int_{\mathbb{R}^3} x_i x_j f(r)\, dV. \tag{161}$$

Since f only depends on r, this integral is isotropic. Hence it is a multiple of δ_{ij}, and we only have to find the diagonal components. Contracting with δ_{ij}, $a_{ii} = \int_{\mathbb{R}^3} r^2 f(r)\, dV$, and so

$$a_{ij} = \frac{1}{3}\delta_{ij}\int_{\mathbb{R}^3} x_i x_j f(r)\, dV. \tag{162}$$

7.4 Symmetric and Antisymmetric

A 2-tensor is called *symmetric* if $T_{ij} = T_{ji}$. It is easy to check in the usual way that being symmetric is basis-independent. Likewise, a 2-tensor is *antisymmetric* if $T_{ij} = -T_{ji}$. This is of course also basis-independent. Similarly, if $p > 2$, $T_{i_1 i_2\cdots i_p}$ can be symmetric (or antisymmetric) on two particular indices.

Decomposition A general 2-tensor T_{ij} can be decomposed into isotropic, symmetric traceless and antisymmetric parts as follows: define

$$T = T_{ii}, \quad S_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3}T\delta_{ij}, \quad A_{ij} = \frac{1}{2}(T_{ij} - T_{ji}) \tag{163}$$

It is easy to check that $S_{ij} = S_{ji}$ and $S_{ii} = 0$, and $A_{ij} = -A_{ji}$. We also have

$$\frac{1}{3}T\delta_{ij} + S_{ij} + A_{ij} = T_{ij}, \tag{164}$$

and this is in fact the unique decomposition with these properties.

In three dimensions, we have a correspondence between antisymmetric tensors and vectors by $A_{ij} = \varepsilon_{ijk}\omega_k$, or $\omega_k = \frac{1}{2}\varepsilon_{ijk}A_{ij}$.

Diagonalisation A symmetric 2-tensor can be diagonalised. A shortcut to get to this is to find the eigenvalues of the corresponding matrix, and use that the eigenspaces of a symmetric matrix are orthogonal to write it as a diagonal matrix.[3] The eigenvalues are called the *principal values*, the directions of the eigenvectors the *principal axes*.

7.5 Examples

Inertia In dynamics, we have $L_i = I_{ij}\omega_j$, where $\mathbf{L}$ is the angular momentum, $\boldsymbol{\omega}$ the angular velocity vector, and the *inertia tensor* is:

$$I_{ij} = \int_{\mathbb{R}^3} (r^2 - x_i x_j)\rho(x)\, dV. \tag{165}$$

This is a tensor by the quotient theorem. The expression follows from $\mathbf{L} = m\mathbf{x}\times\mathbf{v} = m\mathbf{x}\times(\boldsymbol{\omega}\times\mathbf{x})$.

Conductivity In a complicated anisotropic material, the current may not flow in the same direction as the electric field, but there is still a linear relationship, given by

$$J_i = \sigma_{ij}E_j \quad \text{or} \quad E_i = \rho_{ij}J_j, \tag{166}$$

where $\mathbf{J}$ is the current, $\mathbf{E}$ the electric field, σ_{ij} the *conductivity tensor*, and ρ_{ij} the *resistivity tensor*. Their matrices are inverses.

Stress and Strain In material physics, *strain* is a measure of the displacements in a body: if the displacement at $\mathbf{x}$ is $\mathbf{u}$, then the *strain tensor* is

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{167}$$

The *stress tensor* p_{ij} gives the j-component of the force along axis i. Hooke's law suggests a linear relationship between stress and strain, which in its most general form is given by

$$p_{ij} = k_{ijkl}e_{kl}, \tag{168}$$

k_{ijkl} again a tensor by the quotient theorem.

[3] But tensors do not really have eigenvalues. Linear Algebra will have the proper proof.