Examples of Things That Are, and Are Not, Vector Spaces

Richard Chapling

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Vectors and Matrices is sometimes so enamoured by $\mathbb{R}^n$ that it forgets to point out that its results apply to plenty of other objects. In this handout we give a wide variety of examples.

1 The axioms

First, we recall the axioms that an object must satisfy in order to be a vector space. A reminder that "$\forall u \in V$" means the following statement is true for every u that is an element of V, while $\exists v \in V$ means that there is an element of V for which the following statement is true.

A vector space over a field F is a set V equipped with two operations: $+ \colon V \times V \to V$ and $. \colon F \times V \to V$, that satisfy

Closure under + $\forall u, v \in V$, $u + v \in V$.[1]

Associativity $\forall u, v, w \in V$,

$$u + (v + w) = (u + v) + w$$

Zero vector $\exists 0 \in V$ such that $\forall u \in V$,

$$0 + u = u + 0 = u$$

Additive inverse $\forall u \in V$, $\exists (-u) \in V$ so that

$$u + (-u) = (-u) + u = 0$$

Commutativity $\forall u, v \in V$,

$$u + v = v + u$$

These five axioms say that + makes V into an abelian group.

Closure under . $\forall \lambda \in F, \forall u \in V, \lambda.u \in V$

Compatibility of field and scalar multiplication $\forall \lambda, \mu \in F, \forall u \in V$,

$$\lambda.(\mu.u) = (\lambda\mu).u$$

Scalar multiplication by identity $\forall u \in V$,

$$1.u = u$$

[1] The closure axioms are sometimes regarded as part of the definition of the operations, rather than a property of them. In my opinion, there's no harm in reinforcing it, even if it is trivially true.

Distributivity of + over . $\forall \lambda, \mu \in F, \forall u \in V$,

$$(\lambda + \mu).u = \lambda.u \boldsymbol{+} \mu.u$$

Distributivity of . over + $\forall \lambda \in F, \forall u, v \in V$,

$$\lambda(u \boldsymbol{+} v) = \lambda.u \boldsymbol{+} \lambda.v$$

These axioms say that the field F *acts* on V. with the scalar multiplication

2 Examples of vector spaces

The trivial space $\{0\}$ is a vector space over any field, called the trivial vector space. (There is only one possible way to define the operations: $0 \boldsymbol{+} 0 = 0$ and $\lambda.0 = 0$ for any $\lambda \in F$.)

The most important spaces F^n, the set of n-tuples of elements of F, (that is, $u = (u_1, u_2, \dots, u_n)$, where $u_i \in F$ for each $i \in \{1, 2, \dots, n\}$) is a vector space over F when given the operations defined "pointwise":

$$u \boldsymbol{+} v = (u_1, \dots, u_n) + (v_1, \dots, u_n) = (u_1 + v_1, \dots, u_n + v_n)$$

and

$$\lambda.u = \lambda(u_1, \dots, u_n) = (\lambda u_1, \dots, \lambda u_n).$$

These vector spaces are very useful, because we can show that any vector space over F of dimension n is isomorphic to F^n.[2]

We also mention that $\mathbb{C}$ is a vector space over $\mathbb{R}$. It turns out that as a vector space, $\mathbb{C}$ is essentially identical to $\mathbb{R}^2$.

Linear maps Let V, W be vector spaces over the same field F. A map $T \colon V \to W$ is called *linear* if $\forall \lambda, \mu \in F$ and $\forall u, v \in V$,

$$T(\lambda.u + \mu.v) = \lambda.T(u) + \mu.T(v)$$

(the operations on the left being V's, on the right $W's$).

The set of such maps, denoted $L(V, W)$, (or if $W = V$, we write $L(V)$ for $L(V, V)$) can be made into a vector space with the operations defined pointwise on V, namely $\forall S, T \in L(V, W)$, $\forall \lambda \in F$ and $\forall u \in V$,

$$(S \boldsymbol{+} T)(u) = S(u) + T(u), \qquad (\lambda.T)(u) = \lambda(T(u)).$$

Here, we have put brackets in places to distinguish the groupings, and write the operations $\boldsymbol{+}$ and . on $L(V, W)$ distinct from those on V. Linear maps are often written without brackets: Tu instead of $T(u)$.

In particular, F is a vector space over F, so the set of linear maps $V \to F$ is a vector space. This object is called the *dual space* of V, denoted $L(V, F) = V^*$, and its objects are variously called *linear functionals, (linear) one-forms* or *covectors.*

[2] This is why we can use column vectors and matrices: $m \times n$ matrices are exactly the linear maps $F^n \to F^m$.

Free vector space Lastly, we come to the most general way possible of making a vector space: take a set S, and form all finite linear combinations of objects in it with coefficients in F:

$$a_{i_1} s_{i_1} + \lambda_{i_2} s_{i_2} + \cdots + a_{i_k} s_{i_k}$$

This object is called the *free vector space* of S over F. We turn it into a vector space similarly to polynomials, sequences or functions, by adding together the coefficients of the same elements of S, and multiplying every coefficient for scalar multiplication.

The free vector space tends to be huge, but has many useful smaller spaces based on it, by imposing relations between the s_i.

2.1 Weird examples

While every finite-dimensional vector space being isomorphic to F^n, this does not mean that they will all look similar, especially to the untrained eye.

Vector addition and scalar multiplication don't have to be addition and multiplication Let $V = \{x \in \mathbb{R} : x > 0\}$. Then V is a vector space over $\mathbb{R}$ when equipped with the operations

$$u \mathbin{\boldsymbol{+}} v = uv, \qquad \lambda.u = u^\lambda.$$

But the zero vector is certainly not 0! In fact, it's 1: $1 \mathbin{\boldsymbol{+}} u = 1u = u = u1 = u \mathbin{\boldsymbol{+}} 1$.

The zero vector is not always 0 Let V be a vector space with addition $+$ and scalar multiplication $.$. Pick a fixed element $w \in V$, and define a new addition and scalar multiplication by

$$u \mathbin{\boldsymbol{+}} v = u + v - w, \qquad \lambda \odot u = \lambda.(u - w) + w$$

Then V is still a vector space over F with these operations! (What is the zero vector?)

A geometrically defined vector space Let V be the set of arrows with their tails at $0 \in \mathbb{R}^n$ and their head at a point of $\mathbb{R}^n$. Define $u \mathbin{\boldsymbol{+}} v$ as the arrow with its tail at the origin and its head at the point where the head of u ends up after translating it so that its tail lies at the head of v. Define $\lambda.u$ as the vector that points in the same direction as u and has length λ times the length of u. Then V is a vector space over $\mathbb{R}$, although it's a pain to prove this.

A non-numeric vector space We recall that the *symmetric difference* of two sets, $A \triangle B$ is the set of elements that are in A and B, but not both; in other words,

$$A \triangle B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

Notice that this operator is commutative, and less obviously, associative: in terms of indicator functions,

$$\chi_{A \triangle B} = \chi_A + \chi_B - 2\chi_A \chi_B,$$

and so

$$\begin{aligned}\chi_{(A \triangle B) \triangle C} &= \chi_{A \triangle B} + \chi_C - 2\chi_{A \triangle B}\chi_C = \chi_A + \chi_B - 2\chi_A\chi_B + \chi_C - 2(\chi_A + \chi_B - 2\chi_A\chi_B)\chi_C \\ &= \chi_A + \chi_B + \chi_C - 2(\chi_A\chi_B + \chi_B\chi_C + \chi_C\chi_A) + 4\chi_A\chi_B\chi_C.\end{aligned}$$

Whatever the set this function represents,[3] it is symmetric in the three original sets, and so the operation must be associative.

We also have the relations

$$A \triangle \emptyset = A \qquad \text{and} \qquad A \triangle A = \emptyset,$$

so given a set Ω, the set of subsets of Ω (often written as 2^Ω) is an abelian group when equipped with the symmetric difference: the identity is the empty set, and every $A \in 2^\Omega$ is self-inverse.

Since every element of this group has order at most 2, we can turn it into a vector space over the field with two elements,[4] by setting

$$0.A = \emptyset, \qquad 1.A = A.$$

In fact, the latter is forced by one of the axioms, the former follows since $0.A = (1+1).A = A \triangle A = \emptyset$, so there is no need to specify either of these: the scalar multiplication is entirely determined by the axioms.

2.2 Infinite-dimensional examples

Sequences A *sequence* in F is a list of elements of F, or more formally, a function $\mathbb{N} \to F$.[5] A sequence is normally written

$$(a_n)_{n\in\mathbb{N}} \qquad \text{or} \qquad (a_n)_{n=0}^{\infty} \qquad \text{or simply} \qquad (a_n),$$

and sometimes even $a(n)$ or just a.

We can then define operations pointwise:

$$(a_n) + (b_n) = (a_n + b_n) \qquad \lambda(a_n) = (\lambda a_n).$$

With these operations, the following are all vector spaces:

- The set of all sequences
- The set of all *eventually constant* sequences (a sequence is eventually constant if there is an N and an $a \in F$ so that $a_n = a$ for every $n > N$.)
- The set of all *eventually zero* sequences
- The set of all convergent sequences (*convergent* will be defined in Numbers and Sets)
- The set of all sequences that converge to 0

[3] It happens to be the set of elements that are in one or three of the sets, but not two.
[4] This is the set $\{0, 1\}$ with addition and multiplication modulo 2.
[5] Sequences can be defined in any set, but we need to add and scalar multiply, so let's stick to F.

Polynomials and power series Given a field, a *polynomial* in X over F is a finite formal linear combination

$$P = a_0 + a_1 X + \cdots + a_k X^k,$$

where X is an *indeterminant*: it is not evaluated, and not expected to live in the field.[6] "Formal" means that this defines the addition, which has no other properties: this sum is not evaluated to a specific value, it simply exists as a set of slots into which the coefficients fit.

We equip this set with termwise operations,

$$\begin{aligned} P + Q &= (a_0 + a_1 X + \cdots + a_k X^k) + (b_0 + b_1 X + \cdots + b_k X^m) \\ &= (a_0 + b_0) + (a_1 + b_1) X + \cdots + (a_{\max\{k,m\}} + b_{\max\{k,m\}}) X^{\max\{k,m\}} \\ \lambda P &= \lambda(a_0 + a_1 X + \cdots + a_k X^k) = (\lambda a_0) + (\lambda a_1) X + \cdots + (\lambda a_k) X^k \end{aligned}$$

where a_i and b_i are set to 0 if $i > k$ or $i > m$ respectively.

The power k on the highest term is called the *degree* of the polynomial. (It is conventional to assign the polynomial 0 the degree $-\infty$, for reasons relating to the multiplication operation you will put on polynomials in IB Groups, Rings and Modules.) Then with the termwise operations the following are vector spaces:

- The set of polynomials of degree at most n
- The set of all polynomials (written $F[X]$)
- The set of all polynomials where the coefficient of X^0 is 0

Similarly, the infinite formal linear combinations

$$P = \sum_{n=0}^{\infty} a_n X^n$$

form a vector space with termwise operations: this is the *vector space of formal power series*, written $F[[X]]$.

A really large infinite vector space Let S be any set, and let $f : S \to F$. The set of all such functions, often written F^S, once again equipped with pointwise operations

$$(f + g)(s) = f(s) + g(s), \qquad (\lambda f)(s) = \lambda f(s),$$

is a vector space. This set can be (almost) as large as we like.

If the set S has more structure, we can often define more restricted classes of functions on it. For example, if $S = F = \mathbb{R}$, the following are vector subspaces of $\mathbb{R}^{\mathbb{R}}$:

- The continuous functions $C(\mathbb{R})$
- The n-times continuously differentiable functions $C^n(\mathbb{R})$
- The functions that are zero outside a finite interval

[6] Polynomials are not to be confused with *polynomial functions*, which are actual functions which take values in the field. For infinite fields, there often appears to not be much difference, but there certainly is in finite fields.

- The functions that have limits as $x \to \pm\infty$
- The functions with period 1 (that is, $\forall x \in \mathbb{R}$, $f(x) = f(x+1)$)
- The functions with period q for some rational q
- Càdlàg[7] functions: functions that are continuous from the right, and have limits from the left.

$\mathbb{R}$ is worse than you think With the usual operations, $\mathbb{R}$ is a vector space over $\mathbb{Q}$.[8]

A less nasty example: the set $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a vector space.

Finally, the set of numbers algebraic over $\mathbb{Q}$ is a vector space over $\mathbb{Q}$. [9] This is actually quite difficult to prove.[10]

2.3 * Making new vector spaces from old ones

Given two sets, there are various simple ways to combine them to produce a new set, the most important being intersection, union, and Cartesian product. Similarly, for two groups, a third can be produced as an intersection, direct product or quotient. Vector spaces have their own operations corresponding to these ideas: some are essentially the same as one of the above, while some are quite different.

2.3.1 Intersection

If V, W are vector spaces over F, then $V \cap W$ is also a vector space over F.

Beware that if we have bases for V and W, $V \otimes W$ may not include any of the vectors in either basis: for example, if we take V and W subspaces of $\text{span}\{e_1, e_2, e_3\}$ given by

$$V = \text{span}\{e_1, e_2\}, \qquad W = \text{span}\{e_1 + e_3, e_2 + e_3\},$$

then

$$V \cap W = \text{span}\{e_1 - e_2\}.$$

2.3.2 Sum

The union of two vector spaces is not a vector space. Instead, we have two ways to "add" vector spaces to make larger ones.

Direct sum The simplest way of sticking two F-vector spaces V, W together is called the *direct sum*, $V \oplus W$. Its elements are pairs $(u, w) \in V \times W$, and we give it operations in a simple way, just as you might expect:

$$(u, w) + (v, x) = (u + v, w + x), \qquad \lambda(u, w) = (\lambda u, \lambda w).$$

[7] From the French: *continue à droite, limite à gauche.*
[8] Can you find a basis? If not, can you tell anything interesting about what a possible basis would be like?
[9] A number is *algebraic* over a field F if it is a root of a nonzero polynomial function with coefficients in F.
[10] Have a go: note that the hard bit is showing that $u + v$ is algebraic if u and v are.

If $\{b_i\}_{i=1}^n$ is a basis for V and $\{c_j\}_{j=1}^m$ is a basis for W, then $\{(b_i, 0)\}_{i=1}^n \cup \{(0, c_j)\}_{j=1}^m$ turns out to be a basis for $V \oplus W$, so the direct sum has dimension $m + n$:

$$\dim(V \oplus W) = \dim V + \dim W.$$

This idea can be extended to an indexed set of vector spaces, $\bigoplus_{i \in I} V_i$, the elements of which are n-tuples in $\bigtimes_{i \in I} V_i$. It remains true that

$$\dim\left(\bigoplus_{i \in I} V_i\right) = \sum_{i \in I} \dim V_i.$$

We are actually familiar with a simple example of the direct sum: as a vector space,

$$F^n = \underbrace{F \oplus \cdots \oplus F}_{n \text{ times}}.$$

Sum In contrast with the direct sum, the (ordinary) *sum* of two vector spaces is the set of linear combinations of elements of both of them: that is,

$$V + W = \{x : x = v + w, \text{ where } v \in V \text{ and } w \in W\}.$$

If $V \cap W$ is nontrivial, an element of $V + W$ can be expressed as such a sum in more than one way: if $y \in V \cap W$, then

$$x = v + w = (v + y) + (w - y)$$

are two possible representations. Obviously we want to say that if $x_1 = v_1 + w_1$ and $x_2 = v_2 + w_2$, then

$$x_1 + x_2 = (v_1 + v_2) + (w_1 + w_2),$$

and for $\lambda \in F$,

$$\lambda x = (\lambda v) + (\lambda w),$$

but it is necessary to check that the addition (and the scalar multiplication) are *well-defined*: that is, the output is independent of the representations of the inputs.

The good news is that if V has a basis B and W has a basis C, then $V + W = \operatorname{span}(B \cup C)$. The bad news is that if the intersection is nontrivial, the representations of elements in terms of $B \cup C$ are not unique, it is not a basis. Thus the dimension of a sum is no longer just the sum of the dimensions: instead, we have

$$\dim(V + W) + \dim(V \cap W) = \dim V + \dim W$$

(Pick a basis for the smallest space, extend it to a bases of each intermediate space, and then show that the union is a basis for the largest space.)

One of the most common misconceptions in Mathematics is that this formula will extend to three vector spaces in the natural way,

$$\dim(U+V+W) \stackrel{?}{=} \dim U + \dim V + \dim W - \dim(U \cap V) - \dim(V \cap W) - \dim(W \cap U) + \dim(U \cap V \cap W)$$

But it doesn't! Some simple counterexamples are available in $F^3 = \operatorname{span}\{e_1, e_2, e_3\}$: let $U = \operatorname{span}\{e_1\}$, $V = \operatorname{span}\{e_2\}$, and suppose that $W = \operatorname{span}\{w\}$ for some $w \in F^3 \setminus (U \cup V)$. Then all the intersections are 0-dimensional, so the right-hand side is

$$1 + 1 + 1 - 0 - 0 - 0 + 0 = 3,$$

but if $w = e_1 + e_2$, for example, the left-hand side is 2.

A final contrast between the sum and the direct sum is illustrated by the example of equal spaces: $V + V = V$, whereas $V \oplus V$ has dimension $2 \dim V$.

The other extreme is when $U \cap V = \{0\}$: then $U + V \cong U \oplus V$, and the isomorphism is natural in that we don't have to make any choices. In such a situation, it is normal to simply declare that the two spaces are *equal*.

2.3.3 Products

Tensor product We would like a method of forming a "product" of vector spaces, in the same generality as the direct sum works for adding spaces together. The way to do this turns out to be the *tensor product*.

Our initial guess would be that the elements are "products" of the elements of V and W. But it is easy to see that such things won't form a vector space. To get around this, we can take the set of all finite linear combinations of such elements: but we already know what this space is, namely the free vector space on $V \times W$:

$$\sum_i \lambda_i (v_i, w_i).$$

But this space will be enormous, because we have numerous different elements that the free vector space says are different, but we feel *shouldn't* be different in a product: generally, we expect products to be distributive, that is, we ought to have

$$(u, w) + (v, w) = (u + v, w) \qquad \text{and} \qquad (u, v) + (u, w) = (u, v + w).$$

Since this tells us that

$$(2u, w) = (u + u, w) = (u, w) + (u, w) = 2(u, w) = (u, w) + (u, w) = (u, 2w)$$

and so on, we should probably also extend this scaling identification to multiplication on the whole field:

$$\lambda(u, w) = (\lambda u, w) = (u, \lambda w).$$

These two properties force the product to be *bilinear*. This has reduced the space to a manageable size and given its products a sensible interpretation, so we can now make the definition:

Let V, W be vector spaces over F. The *tensor product* of V and W, denoted $V \otimes W$, is the set of all finite linear combinations of objects of the form $v \otimes w$, where $v \in V$ and $w \in W$, subject to the relations

$$\begin{aligned} (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\ \lambda(v \otimes w) &= (\lambda v) \otimes w = v \otimes (\lambda w) \end{aligned}$$

for every $v, v_1, v_2 \in V$, for every $w, w_1, w_2 \in W$ and for every $\lambda \in F$.

The tensor product is a vector space. If $\{b_i\}_{i=1}^n$ is a basis for V and $\{c_j\}_{j=1}^m$ is a basis for W, then it turns out that $\{b_i \otimes c_j\}_{i=1,j=1}^{n,m}$ is a basis for $V \otimes W$, so the tensor product has dimension mn:

$$\dim(V \otimes W) = (\dim V)(\dim W).$$

Exterior product If $V = W$ in the tensor product, we can define other objects by imposing symmetry conditions on the product. The *exterior product* is one such: we define a product $\wedge$ so that it is bilinear like the tensor product, but unlike the tensor product, we also impose the *alternating* condition $v \wedge v = 0$ for every $v \in V$. This and bilinearity forces the product to be antisymmetric: $u \wedge v = -v \wedge u$. The resulting space is normally denoted $\Lambda^2 V$.

In the same way, we can define the multiple exterior product: $\Lambda^k V$ is the set of all objects $v_1 \wedge v_2 \wedge \cdots \wedge v_k$, where the product is linear in every argument, and zero when an argument is repeated; this is again called alternating. By the same argument, the product is *totally antisymmetric*: interchanging any two arguments multiplies the wedge product by -1. The set of all linear combinations of such objects is denoted $\Lambda^k V$. Conventionally, $\Lambda^0 V = F$ and $\Lambda^1 V = V$.

What is the dimension of $\Lambda^k V$? If $\{b_i\}_{i=1}^n$ is a basis for V, any subset of the b_i of size k produces a product $b_{i_1} \wedge b_{i_2} \wedge \cdots \wedge b_{i_k}$, and it turns out that any element of $\Lambda^k V$ can be expanded in terms of these (expand each vector in each wedge product in terms of the b_i). Hence these products form a basis. But there are $\binom{n}{k}$ subsets of size k of a set of size n, so

$$\dim(\Lambda^k V) = \binom{n}{k}.$$

However, this means that $\Lambda^n V$ is 1-dimensional, and all larger exterior powers are 0-dimensional (and hence trivial).

Exterior algebra The *exterior algebra* ΛV is the vector space made by taking the direct sum of all the multiple exterior products of V,

$$\Lambda V = F \oplus V \oplus \Lambda^2 V \oplus \cdots \oplus \Lambda^n V = \bigoplus_{k=0}^{n} \Lambda^k V$$

There are other useful parts of the tensor product, such as the *symmetric product*, but they are rather less important and we won't discuss them now.

2.3.4 Subspaces

Let V be a vector space over F. A *subspace* of V is a subset W of V that is also a vector space (over F). We write $W \leqslant V$.

Examples

- For any vector space V, $V \leqslant V$ and $\{0\} \leqslant V$. A subspace that is not either of these is called *proper*.
- For any vector spaces V, W, $V \cap W \leqslant V \leqslant V + W$

- If $B \subseteq A$, span $B \leqslant$ span A
- $\mathbb{Q} \leqslant \mathbb{R}$
- $C^n(\mathbb{R}) \leqslant \cdots \leqslant C^1(\mathbb{R}) \leqslant C(\mathbb{R}) \leqslant \mathbb{R}^{\mathbb{R}}$
- For real sequences,

$$\{\text{eventually zero}\} \begin{matrix} \leqslant \{\text{eventually constant}\} \leqslant \\ \leqslant \{\text{converging to } 0\} \leqslant \end{matrix} \{\text{convergent}\} \leqslant \{\text{all real sequences}\},$$

neither of the stacked ones being a subspace of the other.

2.3.5 Quotients

The sum and product generally involve making a larger space out of smaller ones. We now consider how to make a vector space smaller, in a different way from intersecting it with things.

This works in a similar way to groups: instead of a normal subgroup, we have a subspace; since the operations on a vector space are commutative, there is no need for a further restriction analogous to "normal".

Let V be a vector space over F, W a subspace of V. We define a relation

$$u \sim v \iff u - v \in W.$$

This turns out to be an equivalence relation[11] The *quotient space* V/W is the set of equivalence classes of this relation:

$$V/W = \{u + W : u \in V\},$$

where $u + W = v + W$ if and only if $u - v \in W$.[12] We equip it with the operations

$$(u + W) + (v + W) = (u + v) + W \qquad \lambda(u + W) = \lambda u + W,$$

and just as one checks for quotient groups and the sum, these operations are well-defined and make V/W into a vector space.

Some examples:

- $V/\{0\} \cong V$ and $V/V \cong \{0\}$ in a natural way.
- $(U + V)/V \cong U/(U \cap V)$ in a natural way.
- As a special case, $(U \oplus V)/V \cong U$ in a natural way.
- $U + V)/(U \cap V) \cong U \oplus V$.

The dimension of V/W is, not surprisingly, the difference between the dimensions:

$$\dim V = \dim W + \dim(V/W)$$

(The easiest way is probably to write V as a direct sum of W and V/W, using bases if necessary. Alternatively, it follows by applying the rank–nullity theorem to the quotient map $u \mapsto u + W$.)

[11] $u - u = 0 \in W$, $u - v \in W \iff v - u = -(u - v) \in W$, and if $u - v \in W$ and $v - x \in W$, then $u - x = (u - v) + (v - x) \in W$, since W is a subspace.

[12] We can think of these objects as slicing V into parallel copies of W.

3 Examples that are not vector spaces

The axioms are not independent: for example, one cannot have additive inverses without first having an identity. It is therefore not always possible to give examples that satisfy all but one of the axioms.[13, 14] Most of these examples work for all fields, but sometimes we need $1 + 1 \neq 0$, i.e. a field of characteristic not equal to 2.

3.1 Not closed under addition

- The set of odd integers with the usual addition (sum of two odd numbers is even)
- The set of vectors in F^2 of the form $(\lambda, 0)$ or $(0, \mu)$, with entrywise addition (scalar multiplication works fine apart from distribution over addition).
- The set of non-convergent sequences.
- The set of all sequences that converge to $a \neq 0$.
- The set of all polynomials over F in X of degree n.

3.2 Not associative

- Consider the addition on F given by $u \boldsymbol{+} v = 2u + 2v$. Then

$$(u \boldsymbol{+} v) \boldsymbol{+} w = 2(u \boldsymbol{+} v) + 2w = 4u + 4v + 2w \neq 2u + 4v + 4w = u \boldsymbol{+} (v \boldsymbol{+} w).$$

3.3 No identity

- The set of positive integers, $\{1, 2, \dots\}$ with the usual addition.
- The set of positive reals with the usual addition.
- The set of all polynomials over F in X of degree n.
- The empty set. (All the other axioms are satisfied vacuously.[15])

3.4 Lack of inverses

- The nonnegative integers or reals with the usual addition: $-1 \notin \mathbb{N}$ and $\notin \mathbb{R}_{\geqslant 0}$.
- $\mathbb{R}$ with the binary operation $x \boldsymbol{+} y = x + y - xy$: $\boldsymbol{+}$ is associative, closed and has identity 0, but 1 has no inverse because $1 \boldsymbol{+} y = 0$, for any y.

[13] Independent sets of axioms do exist: see for example Rigby, J.F. and Wiegold, James, "Independent axioms for vector spaces", *The Mathematical Gazette*, Vol. 57, No. 399 (1973), pp. 56–62. https://doi.org/10.1017/S0025557200131651

[14] One can also make our 8 non-closure axioms independent by replacing the existence of inverses with right cancellation, and then it is possible to give examples that violate only one of the axioms: https://math.stackexchange.com/a/24880/221811 .

[15] A statement is *vacuously true* if it is about objects of the empty set (so there are no examples to "stop" it being true). You can say whatever you like about the largest natural number, apart from that it exists.

3.5 Not abelian

- Any nonabelian group will do.
- $\mathbb{R}$ with $u \mathbin{\mathbf{+}} v = 2u + v$, which is also not associative.
- $\mathbb{R}$ with $u \mathbin{\mathbf{+}} v = 0$ for every $u, v \in \mathbb{R}$ is associative, but also has no inverses.

3.6 Not closed under scalar multiplication

- $\mathbb{Z}$ is not a vector space over $\mathbb{Q}$ or $\mathbb{R}$, because $\frac{1}{2}1 \notin \mathbb{Z}$.

3.7 Incompatible scalar multiplication

- Take F equipped with the scalar multiplication $\lambda.u = -\lambda u$. Then

$$(\lambda\mu).u = -\lambda\mu u \neq \lambda\mu u = \lambda.(\mu.u)$$

3.8 Field identity behaves wrongly

- For any F-vector space V, defining $\lambda u = 0$ for every $u \in V$ and $\lambda \in F$ satisfies every other axiom.

3.9 Not distributive

- Taking F equipped with the scalar multiplication $\lambda.u = \lambda^2 u$, then

$$(1+1).u = 2.u = 4u \neq 1.u + 1.u = u + u = 2u$$

  (but all the other axioms are satsified).

- Violating the other distributivity axiom while satisfying all the other axioms is actually quite difficult.[16] A somewhat manufactured example for a complex field is the multiplication

$$\lambda.(a,b) = \begin{cases} (\lambda a, \lambda b) & a \neq 0 \\ (0, \bar{\lambda} b) & a = 0 \end{cases},$$

  where $\bar{\lambda}$ is the complex conjugate of λ.

[16] How much distributivity can you get with only the other axioms?