VP Sheet 1 Question 9: Minimal Surface of Revolution Bounded by Two Circles

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We want to find a surface of revolution of minimal area that has as its boundary the circles $r = a$, $z = \pm b$.

The area of a surface of revolution, written in cylindricals as the graph of $r = r(z)$, is given by

$$A[r] = \int_{z_1}^{z_2} 2\pi r\sqrt{1 + r'^2}\, dz$$

(consider it as made of parallel slices of width δz, which are uniformly approximated by frusta of average radius $2\pi r$ and slant height equal to the length of the curve, i.e. $\sqrt{\delta z^2 + \delta r^2} \approx \sqrt{1 + r'(z)^2}\,\delta z$ and so on).

Form of solutions To minimise the surface area, we therefore consider the Lagrangian

$$L[r] = r\sqrt{1 + r'^2}.$$

This is independent of the independent variable z, so we apply the Beltrami identity

$$r'\frac{\partial L}{\partial r'} - L = C,$$

a constant. We thus find that

$$C = \frac{r'rr'}{\sqrt{1 + r'^2}} - \sqrt{1 + r'^2} = \frac{rr'^2}{\sqrt{1 + r'^2}} - \frac{r(1 + r'^2)}{\sqrt{1 + r'^2}} = \frac{r}{\sqrt{1 + r'^2}}.$$

Rearranging this, we obtain the integrable form

$$\pm 1 = \frac{Cr'}{\sqrt{r^2 - C^2}}.$$

Putting $r = C\cosh(u/C)$, $r' = u'\sinh(u/C)$ and the differential equation simplifies to $u' = \pm 1$, so we obtain $u = \pm(z - z_0)$, and

$$r = C\cosh\big(\pm(z - z_0)/C\big).$$

The $\pm$ is redundant, and we have $\cosh((a - z_0)/C) = b/C = \cosh((-a - z_0)/C)$, so $z_0 = 0$, and we obtain the required form

$$r = c\cosh(z/c). \tag{1}$$

This is called a *catenary*, and gives rise to surface of revolution called a *catenoid*. To satisfy the boundary conditions, we need the points (b, a) to lie on the curve, i.e. $r(b) = a$. This gives the relation that c must satisfy,

$$a/c = \cosh(b/c). \tag{2}$$

Since r is positive, c must be positive. How many positive c satisfy (2), and which, if any of these solutions are the global minimiser (at least among surfaces of revolution)?

Notice that the problem is scale-invariant, in the sense that scaling a, b, c (and hence also r and z) by the same constant does not change the ratios a/c, b/c, or r/c and z/c on which the solution depends. This is reflected in Figure 2a, where we see that the family of curves has a pair of straight lines through the origin as its envelope. Therefore, define

$$\gamma = \frac{c}{a} \quad k = \frac{b}{a} \quad R = \frac{r}{a} \quad Z = \frac{z}{b}$$

as "dimensionless" variables. (1) and (2) become

$$R = \gamma\cosh(kZ/\gamma), \qquad 0 \leqslant R, Z \leqslant 1$$
$$1/\gamma = \cosh(k/\gamma).$$

We then obtain at once an expression for k in terms of γ:

$$k = \gamma \operatorname{arg\,sech}\gamma.$$

Particularly notable from this equation is that there is a maximum possible value of k, which we find is $k_1 = 0.662\ldots$, corresponding to $\gamma = 0.552\ldots$. For $0 < k < k_1$, there are two γ corresponding to each k, and for $k > k_1$, there are no catenoid solutions possible.

$k > k_1$: Goldschmidt solution Physically, one would expect there to be a solution for $k > k_1$, but what is it? The answer is the disconnected surface composed of two flat disks with the circles as their boundaries. This is called the *Goldschmidt solution*, and clearly has area $A_G = 2\pi a^2$. We could never find this solution with our calculus of variations argument, since we assumed that the surface was a (continuous, differentiable) function of z, which this clearly is not. An intuitive argument implies that this solution is a local minimiser of area. It is certainly also better than a catenoid when no catenoid exists, but can it have smaller area than a catenoid?

Area of catenoids We first compute

$$\begin{aligned} A[r] &= 2\pi\int_{-b}^{b} c\cosh(z/c)\sqrt{1 + \sinh^2(z/c)}\, dz \\ &= 2\pi\int_{-b}^{b} c\cosh^2(z/c)\, dz \\ &= 2\pi\int_{-b}^{b} \frac{c}{2}\big(1 + \cosh(2z/c)\big)\, dz \\ &= 2\pi\big(bc + c^2\cosh(b/c)\sinh(b/c)\big). \end{aligned}$$

If we replace b with $c \operatorname{arg\,cosh}(a/c)$, we find that

$$A = 2\pi c^2\Big(\sqrt{(a/c)^2 - 1} + \operatorname{arg\,cosh}(a/c)\Big).$$

Again, it is easier to parametrise the surface in terms of γ, rather than k, so replacing c/a with γ gives

$$A = 2\pi a^2\Big(\gamma\sqrt{1 - \gamma^2} + \gamma^2 \operatorname{arg\,sech}\gamma\Big).$$

Perhaps surprisingly, we find that this function takes its maximum at the same value of γ as k does, namely $0.552\ldots$. (Indeed, we find that $dA/d\gamma = 2\gamma\, dk/d\gamma$, so the derivatives have the same sign for $0 < \gamma < 1$.) This maximum is $(1.199\ldots)A_G$, so there are values of γ (and hence k) where $A > A_G$. Solving numerically, we find that $A = A_G$ for $\gamma_2 = 0.825\ldots$, which corresponds to $k_2 = 0.527\ldots$.

All these results are summarised in the plots in Figure 1.

So therefore:

- For $0 < k < k_2$, there are two catenoid solutions, one with larger area than the Goldschmidt solution, one with smaller area.
- For $k_2 \leqslant k < k_1$, there are two catenoid solutions, both of which have area at least as large as the Goldschmidt solution.
- For $k = k_1$, there is one catenoid solution, with area approximately 1.2 times that of the Goldschmidt solution.
- For $k > k_1$, there is only the Goldschmidt solution.

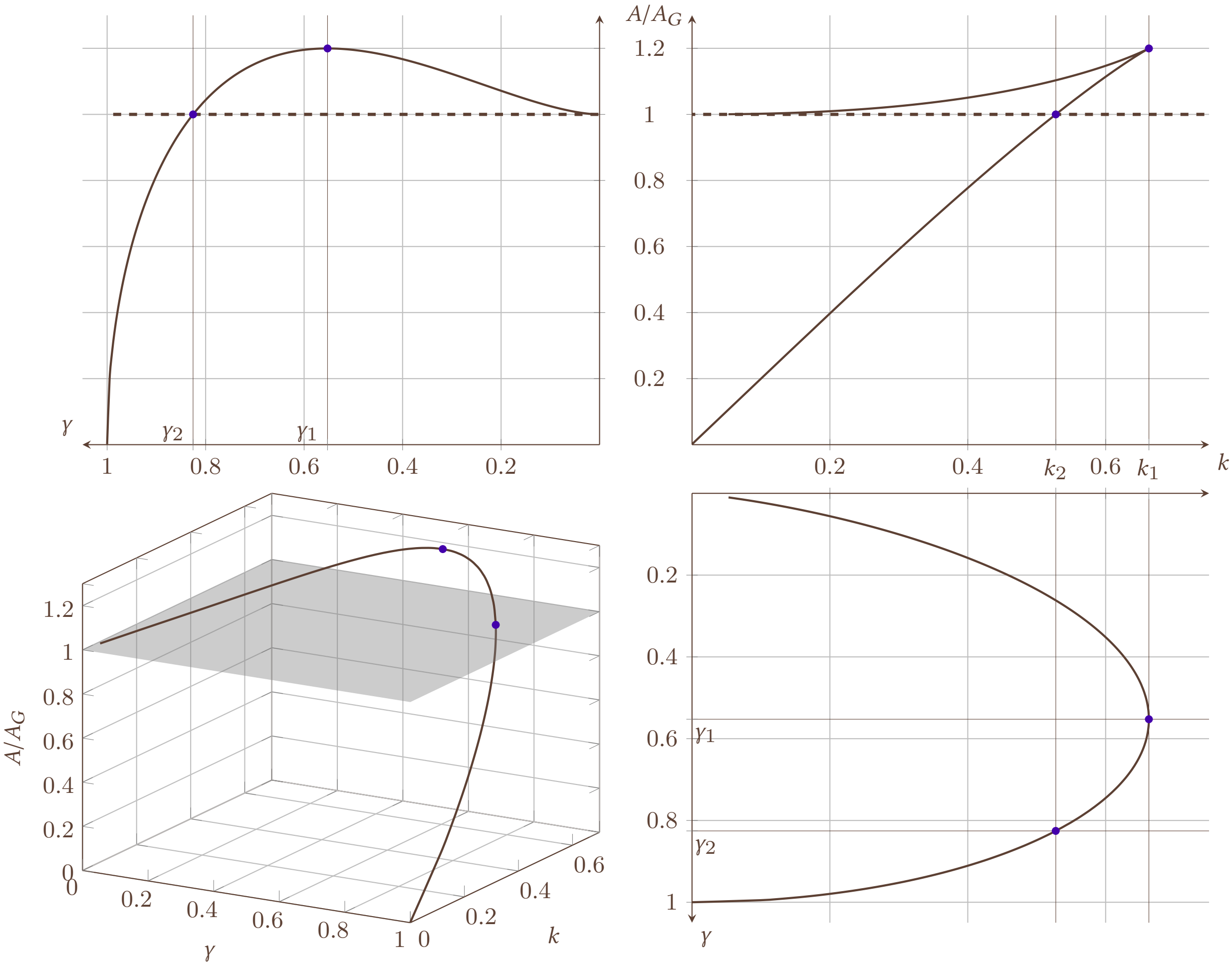
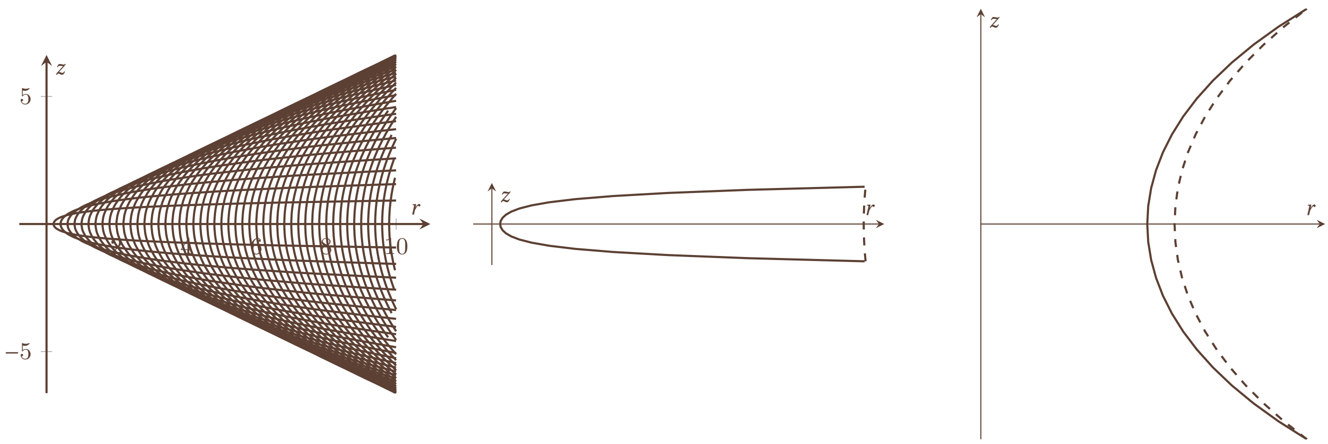


Figure 1: Graphs of γ, k and A/A_G, with k_1, k_2, γ_1, γ_2 marked. Clockwise from top left: A/A_G as a function of γ; A/A_G against k, k as a function of γ; all three together on a 3D plot. The first three images are different views of the fourth.



(a) The family of curves $r/c = \cosh(z/c)$. Scaling properties are easily visible.

(b) Catenaries for $k = 0.1$. One is nearly cylindrical and has small surface area. The other is closer to the two disks, with surface area larger than A_G.

(c) Catenaries for $k = 0.66$. The catenaries become closer together until they merge at $k = k_1$. Both have surface area larger than A_G

Figure 2: The family of catenary solutions, and specific examples