The Γ Function

Properties useful for Asymptotic Methods

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v1 30 November 2024

Definition 1. For all z with $\mathrm{Re}(z) > 0$ the *Gamma-function* is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{1}$$

We basically only care about real z in this course.

1 Elementary formulae

Specific values

$$\Gamma(1) = 1 \tag{2}$$

Proof. Integral is elementary for $z = 1$. □

$$\Gamma(1/2) = \sqrt{\pi} \tag{3}$$

Proof.

$$\Gamma(1/2)^2 = \int_{t=0}^\infty \int_{s=0}^\infty s^{-1/2} t^{-1/2} e^{-s-t}\, ds\, dt$$

Change variables to $s = u(1-v)$, $t = uv$, $ds\, dt = u\, du\, dv$

$$= \int_{u=0}^\infty \int_{v=0}^1 \frac{u^{-1/2-1/2+1}}{\sqrt{v(1-v)}} e^{-u}\, dv\, du$$

$$= \left(\int_0^\infty e^{-u}\, du \right) \left(\int_0^1 \frac{dv}{\sqrt{v(1-v)}} \right)$$

First integral is 1, second substitute $u = \sin^2\theta$

$$= \int_0^{\pi/2} 2\, d\theta = \pi,$$

integrand is positive so take the positive root, done. □

Functional equation

$$z\Gamma(z) = \Gamma(z+1) \tag{4}$$

Proof. Integrate the definition by parts. □

Γ is an extension of the factorial For any $n \in \mathbb{N}$,

$$\Gamma(n+1) = n!. \tag{5}$$

Proof. By induction on n. □

Generalised binomial coefficients For any $z \in \mathbb{C}$ and $|t| < 1$,

$$\begin{aligned}(1+t)^z &= \sum_{k=0}^\infty \binom{z}{k} t^k \\ &= \sum_{k=0}^\infty \frac{z(z-1)\cdots(z-k+1)}{k!} t^k \\ &= \sum_{k=0}^\infty \frac{\Gamma(z+1)}{\Gamma(z-k+1)k!} t^k,\end{aligned} \tag{6}$$

i.e.

$$\binom{z}{k} = \frac{\Gamma(z+1)}{\Gamma(z-k+1)k!} \tag{7}$$

2 Substitutions

Scaling For any $a > 0$,

$$\int_0^\infty t^{z-1} e^{-at}\, dt = \frac{\Gamma(z)}{a^z} \tag{8}$$

(Also the Laplace transform of t^z.)

Proof. Put $t = au$ in the definition. □

Exponentials containing a power For any $a > 0$,

$$\int_0^\infty \exp(-t^a)\, dt = \Gamma\left(1 + \frac{1}{a}\right). \tag{9}$$

Proof. Put $t = u^a$ in the definition. □

Onesided Gaussian moments For any $a > -1$,

$$\int_0^\infty t^a e^{-t^2}\, dt = \frac{1}{2}\Gamma\left(\frac{1+a}{2}\right) \tag{10}$$

Proof. Put $t = u^2$ in the definition. □

Twosided Gaussian moments If n is an even nonnegative integer, we have the formula[1]

$$\int_{-\infty}^\infty t^{2n} e^{-t^2/2}\, dt = 2^{n+1/2}\Gamma\left(n + \frac{1}{2}\right) = (2n-1)!!\sqrt{2\pi}. \tag{11}$$

Proof. Put $t = su^2$ in the definition with $z = 1/2$. Use induction, differentiate n times with respect to s. □

[1] Reminder: the double factorial $n!!$ is defined by $1!! = 2!! = 1$ and $(n+2)!! = (n+2)n!!$. We also take $(-1)!! = 1$, we can extend to further odd negative integers by induction if we so desire.

3 Imaginary integrals

Γ with imaginary exponent If $0 < \operatorname{Re} z < 1$,

$$\int_0^\infty t^{z-1} e^{\pm it}\, dt = e^{\pm i\pi z/2}\Gamma(z) \tag{12}$$

(This is an improper Riemann integral, it cannot be a Lebesgue integral.)

Proof. We prove the positive case. The negative case is exactly the same but we use a semicircle below the real axis. Consider

$$\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} w^{z-1} e^{iw}\, dw,$$

where

- $\gamma_1 = (\varepsilon, R)$,
- $\gamma_2 = (Re^{i\theta} : \theta \in (0, \pi))$, (traversed anticlockwise)
- $\gamma_3 = (iR, i\varepsilon)$,
- $\gamma_4 = \{\varepsilon e^{i\theta} : \theta \in (\pi, 0)\}$ (traversed clockwise).

See Figure 1.

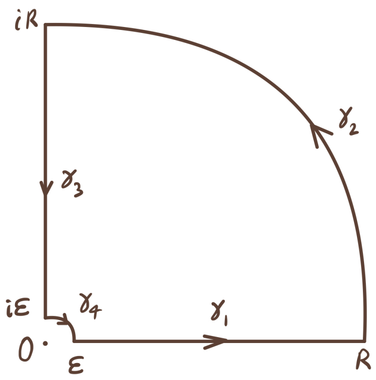

Figure 1: Contour for imaginary Γ

and we follow the branch of w^z that is 1 at $w = 1$.

This integral vanishes by Cauchy's theorem since the function is analytic inside this contour. Now look at the individual contours:

γ_1

$$\int_{\gamma_1} = \int_\varepsilon^R t^{z-1} e^{it}\, dt \to \int_0^\infty t^{z-1} e^{it}\, dt$$

γ_2 By Jordan's Lemma, this converges to 0 as $R \to \infty$ since $\operatorname{Re}(z) < 1$.

γ_3 Substituting $w = e^{i\pi/2}u = iu$, so $dw = i\, du$,[2]

$$\begin{aligned}\int_{\gamma_3} &= \int_{iR}^{i\varepsilon} w^{z-1} e^{iw}\, dw \\ &= \int_R^\varepsilon (ue^{i\pi/2})^{z-1} e^{-u} i\, du \\ &= -e^{i\pi z/2}\int_\varepsilon^R u^{z-1}e^{-u}\, du \to -e^{i\pi z/2}\Gamma(z).\end{aligned}$$

γ_4 We have the bound

$$\left|\int_{\gamma_4}\right| \leqslant \frac{\pi}{4}\varepsilon\varepsilon^{z-1}1 = O(\varepsilon) \to 0,$$

using the "rectangle bound" $|\int_\gamma f| < \ell(\gamma)\sup_\gamma f$.

Hence obtain result. □

Imaginary integral with stationary point For any $a > 1$,

$$\int_0^\infty e^{\pm it^a}\, dt = e^{\pm i\pi/(2a)}\Gamma\left(1 + \frac{1}{a}\right) \tag{13}$$

Proof. Substitute $t = u^a$ in (12). □

4 Miscellaneous

Complex conjugation

$$\Gamma(\bar{z}) = \overline{\Gamma(z)} \tag{14}$$

Reflexion formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{15}$$

Most familiar way to prove this is to use the Beta function and a contour integral.[3] See Further Complex Methods.

Stirling's formula As $x \to +\infty$,

$$\Gamma(x) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + O\left(\frac{1}{x^2}\right)\right). \tag{16}$$

Proof. In lectures/Example sheet 2! □

[2] Reminder: the point of writing it this way is to emphasise how we compute the branch of w^z, by following γ_2 round from the real axis.

[3] And is certainly not relevant for this course!