Why Are Möbius Transformations Like Matrices?

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One of the more puzzling aspects of $\mathcal{M}$, the set of Möbius transformations, is that we are told that we can represent

$$z \mapsto \frac{az+b}{cz+d} \qquad \text{by} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The purpose of this handout is to explain that this is not really mysterious, once you have a gap in your knowledge filled.

1 The Complex Projective Line

Definition 1. The *complex projective line*, $\mathbb{CP}^1$, is defined as the set

$$(\mathbb{C}^2 \setminus \{(0,0)\})/\sim,$$

where $\sim$ is an equivalence relation given by $(z_0, z_1) \sim (w_0, w_1)$ when $z_0 w_1 - z_1 w_0 = 0$.

There are a number of ways to think about this, including lines through the origin in $\mathbb{C}^2$, a sphere in three dimensions, and the one that we are actually interested in: *this is equivalent to the Riemann sphere.*

Relationship to the Riemann sphere Suppose we look at points of the form $(z,1) \in \mathbb{CP}^1$. Then z can be any complex number we like. Therefore the set $\{(z,1) : z \in \mathbb{C}$ is isomorphic to $\mathbb{C}$. Also, for any $z_1 \neq 0$, we have $(z_0, z_1) \sim (z_0/z_1, 1)$, so in fact any point in $\mathbb{CP}^1$ with its second coordinate nonzero can be thought of as within the complex plane. On the other hand, suppose $z_1 = 0$. Then $(z_0, 0) \sim (1, 0)$, so there is only one point of this form, and this becomes the point at infinity, ∞. Therefore we have the decomposition

$$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

Joining in the point at infinity Certainly as a set this looks like the Riemann sphere, but does it join up with the point at infinity in the way we expect? That is, can we use our nice crude limiting definitions of how the point at infinity works?

Convergence to a limit in $\mathbb{C}$ works as before. On the other hand, convergence to ∞ means that $|z|$ becomes larger and larger. In $\mathbb{CP}^1$, this means that the z in $(z,1)$ gets larger and larger. But this point is equivalent to $(1, 1/z)$, which we can see approaches $(1,0)$, which we identified with the point ∞.[1]

*** Note on the topology** The following comments are worth reading after you know how basic topology works (see, for example, IB Analysis and Topology): they provide a more detailed understanding of how the joining works.

The topology of the Riemann sphere is given by taking a basis made out of the balls in the plane, but adjoining sets of the form $\{z \in \mathbb{C} : |z| > r\} \cup \{\infty\}$, for all $r > 0$.

On the other hand, $\mathbb{CP}^1$ inherits the topology of $\mathbb{C}^2$, but with equivalent points identified. The part of this we actually care about is that if we have a ball that does not contain the point at infinity, it is exactly a ball in the $(z,1)$ complex plane. On the other hand, a ball centred at $(1,0)$ is of the form

$$\{(1,z) \in \mathbb{C}^2 : |z| < \varepsilon\},$$

which is equivalent to

$$\{(w,1) \in \mathbb{C}^2 : |w| > 1/\varepsilon\} \cup \{(1,0)\}.$$

It should be clear that these are of the same form as the open sets we add to $\mathbb{C}$ to get the Riemann sphere.

[1] Admittedly I have (at least partially deliberately) blurred the distinction between the copy of $\mathbb{C}$ we can identify in $\mathbb{CP}^1$ and $\mathbb{C}$ itself, but hopefully the main idea is clear.

2 Back to Möbius Transformations

That's all very well, but what does this have to do with Möbius transformations? The invertible linear transformations of $\mathbb{C}^2$ are obviously given by matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc \neq 0$$

But suppose I consider acting on a point (z_0, z_1) where $z_1 \neq 0$ so $(z_0, z_1) \sim (z, 1)$, and suppose it maps to a point where $cz_0 + dz_1$ is not zero. Then

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z, 1) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\frac{z_0}{z_1}, 1 \right) \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_0, z_1) \\ &= (az_0 + bz_1, cz_0 + dz_1) \\ &= \left(\frac{az_0 + bz_1}{cz_0 + bz_1}, 1 \right) \\ &= \left(\frac{az + b}{cz + d}, 1 \right), \end{aligned}$$

which is the same answer as the Möbius transformation acting on z! It follows that, on the complex plane part of $\mathbb{CP}^1$, Möbius transformations and the "projective linear transformations" given by complex 2×2 matrices are identical.

It is left as an exercise to the reader to fiddle through the two other cases: the algebra works in exactly the same way, and you will find that the continuity method we normally use to evaluate at ∞ falls out of this formalism entirely naturally, because we set up limits to work properly!

The projective special linear group Lastly, we find a group of matrices that is in bijection with the Möbius transformations. Notice we can't just take $GL(2, \mathbb{C})$ (all invertible 2×2 complex matrices), because the scaling invariance of the coordinates in the projective line means that A and λA correspond to the same transformation.[2]

Not surprisingly, the solution to this problem is to quotient out by a subgroup, in this case the subgroup $Z = \{\lambda I : \lambda \in \mathbb{C}\}$. It is easy to check that this is normal, since I commutes with everything, and hence we can define the quotient group

$$\mathrm{PGL}(2, \mathbb{C}) := \mathrm{GL}(2, \mathbb{C})/Z.$$

This group is called the *projective general linear group*, and is precisely the group of linear transformations on $\mathbb{CP}^1$.

Our previous work thus shows that

$$\mathcal{M} \cong \mathrm{PGL}(2, \mathbb{C}),$$

the identification we wanted!

[2] This is also apparent in the usual formula for the Möbius transformation, of course, since the λs cancel out in the numerator and denominator.